Chapter 1
Stress

The bolts used for the connections of this steel framework are subjected to stress. In this chapter we will discuss how engineers design these connections and their fasteners.

1.1 Introduction

○ Mechanics of materials – a branch of mechanics

- the relationships between the *external* loads applied to a deformable body and the intensity of *internal* forces acting within the body
- the *deformations* of the body
- the body's *stability* when the body is subjected to external forces

○ Design of structures or machines

- determine the forces acting both on and within its various members by principles of statics
- choose the type of materials and design the size of the members
- check the deflection and stability

✓ Many formulas and rules for design, as defined in engineering codes and used in practice, are based on the fundamentals of mechanics of materials.

✓ An accurate determination and fundamental understanding of material behavior will be of vital importance for developing the necessary equations used in mechanics of materials.

○ Historical Development

- ➢ Early 17th century

  Galileo performed experiments to study the effects of loads on rods and beams made of various materials.

- ➢ Early 18th century

  Saint-Venant, Poisson, Lame, and Navier in France conducted both experimental and theoretical studies on material-body applications of mechanics, called “strength of materials”.

- ➢ Over the years

  After many of the fundamental problems of mechanics of materials had been solved, it became necessary to use advanced mathematical and computer techniques to solve more complex problems.
  – theory of elasticity, theory of plasticity

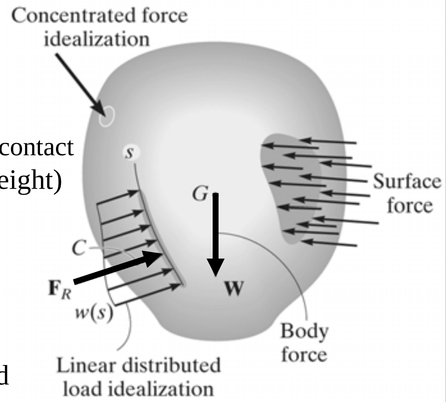
- ➢ Ongoing research

  To meet the demands for solving advanced engineering problems

1.2 Equilibrium of a Deformable Body

○ External Loads

- ➢ Body Forces
  - ✓ without direct physical contact
  - ✓ earth’s gravitation (weight)
  - ✓ electromagnetic field
- ➢ Surface Forces
  - ✓ direct contact
  - ✓ concentrated force
  - ✓ linear distributed load



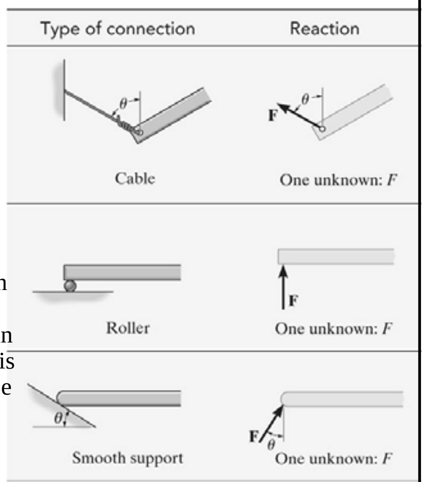
$\mathbf{F}_R$: equivalent to the area under the distributed loading curve, and this resultant acts through the centroid C or geometric center of this area

○ Support Reactions

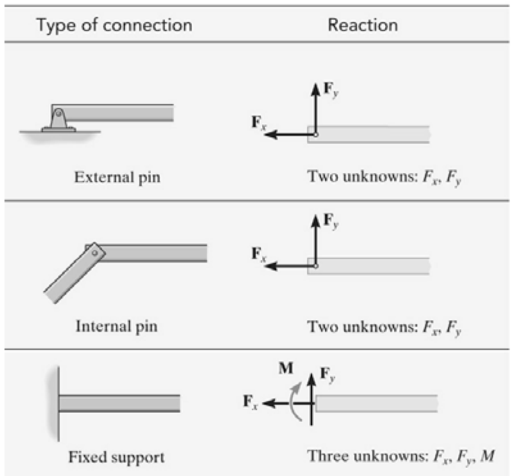
- ➢ Reactions

  the surface forces that develop at the supports or points of contact between bodies

✓ If the support prevents translation in a given direction, then a force must be developed on the member in that direction. Likewise, if rotation is prevented, a couple moment must be exerted on the member.



○ Support Reactions



○ Equations of Equilibrium

Equilibrium of a body requires **both a balance of forces**, to prevent the body from translating or having accelerated motion along a straight or curved path, **and a balance of moments**, to prevent the body from rotating.

in vector form $\boxed{\Sigma \mathbf{F} = \mathbf{0} \quad \Sigma \mathbf{M}_O = \mathbf{0}}$

in scalar form

$$\boxed{\begin{array}{lll} \Sigma F_x = 0 & \Sigma F_y = 0 & \Sigma F_z = 0 \\ \Sigma M_x = 0 & \Sigma M_y = 0 & \Sigma M_z = 0 \end{array}}$$

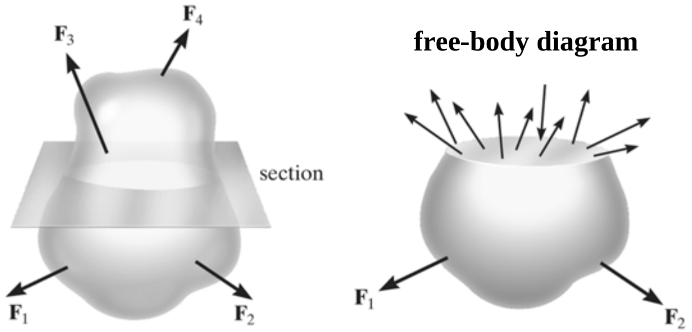
a system of coplanar forces $\boxed{\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma M_O = 0}$

✓ The best way to account for these forces is to draw the body's free-body diagram.

○ Internal Resultant Loadings

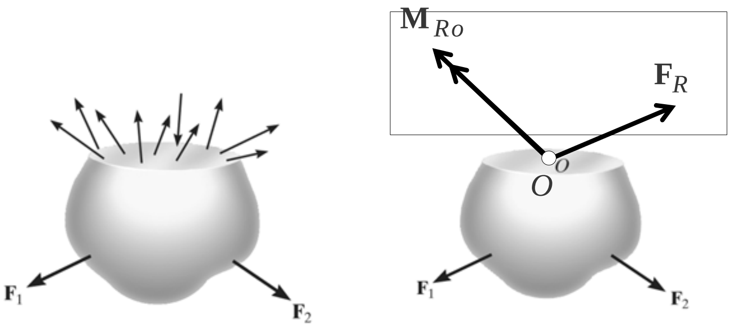
➢ Method of sections

An imaginary section or "cut" is made through the region where the internal loadings are to be determined.



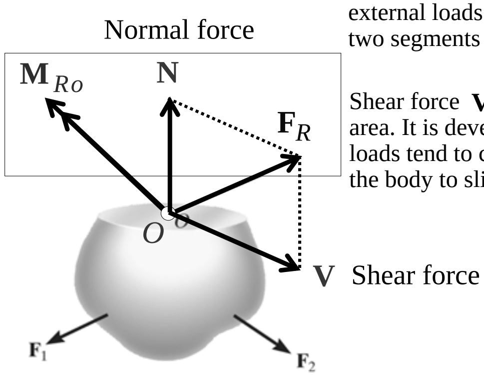
$\mathbf{F}_R$: resultant force at specific point O

$\mathbf{M}_{R_O}$: resultant moment at specific point O



➢ Cross section: the section is taken perpendicular to the longitudinal axis of the member.

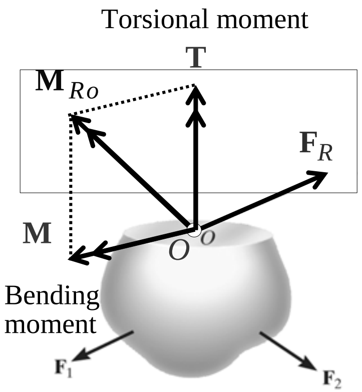
Three Dimensions



Normal force $\mathbf{N}$: acting perpendicular to the area. It is developed whenever the external loads tend to push or pull on the two segments of the body

Shear force $\mathbf{V}$: lying in the plane of the area. It is developed when the external loads tend to cause the two segments of the body to slide over one another

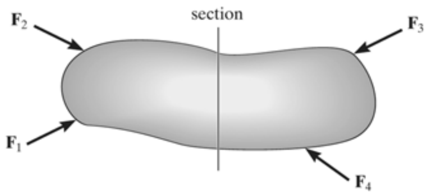
Three Dimensions

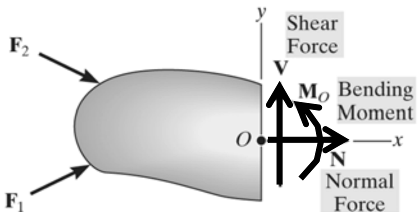


Torsional moment or torque **T** : This effect is developed when the external loads tend to twist one segment of the body with respect to the other.

Bending moment **M** : The bending moment is caused by the external loads that tend to bend the body about an axis lying within the plane of the area

Coplanar Loadings





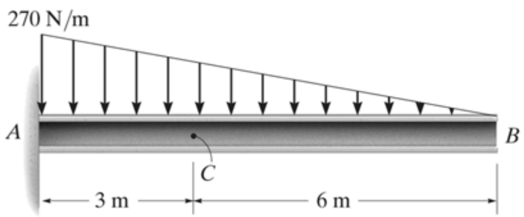
$$\Sigma F_x = 0 \rightarrow \mathbf{N}$$

$$\Sigma F_y = 0 \rightarrow \mathbf{V}$$

$$\Sigma M_o = 0 \rightarrow \mathbf{M}_o$$

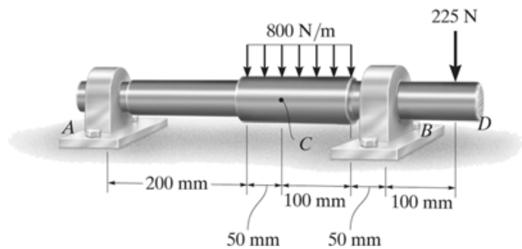
EXAMPLE 1.1

Determine the resultant internal loadings acting on the cross section at *C* of the cantilevered beam shown in Figure.



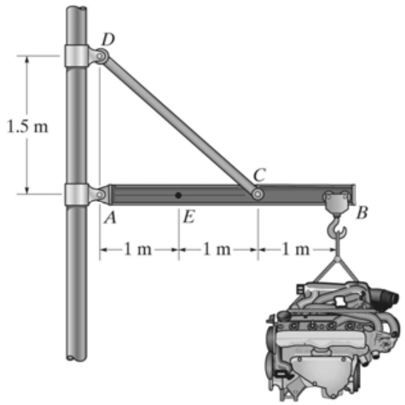
EXAMPLE 1.2

Determine the resultant internal loadings acting on the cross section at *C* of the machine shaft shown in Figure. The shaft is supported by bearings at *A* and *B*, which only exert vertical forces on the shaft.



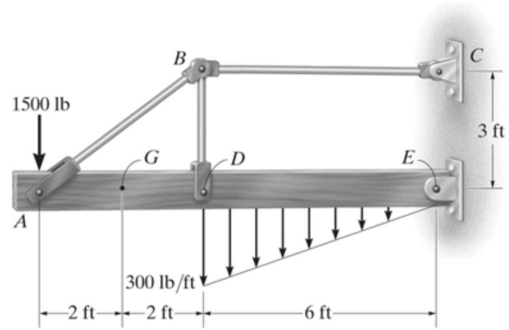
EXAMPLE 1.3

The 500-kg engine is suspended from the crane boom in Figure. Determine the resultant internal loadings acting on the cross section of the boom at point *E*.



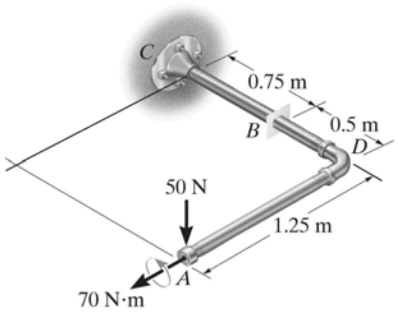
EXAMPLE 1.4

Determine the resultant internal loadings acting on the cross section at *G* of the beam shown in Figure. Each joint is pin connected.



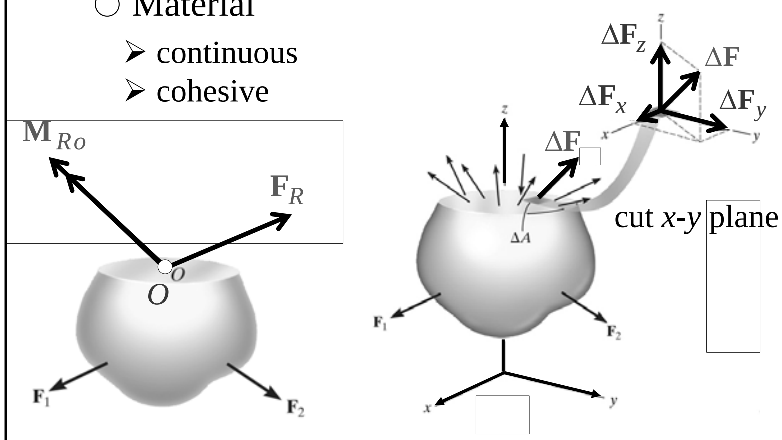
EXAMPLE 1.5

Determine the resultant internal loadings acting on the cross section at *B* of the pipe shown in Figure. The pipe has a mass of 2 kg/m and is subjected to both a vertical force of 50 N and a couple moment of 70 N-m at its end *A*. It is fixed to the wall at *C*.

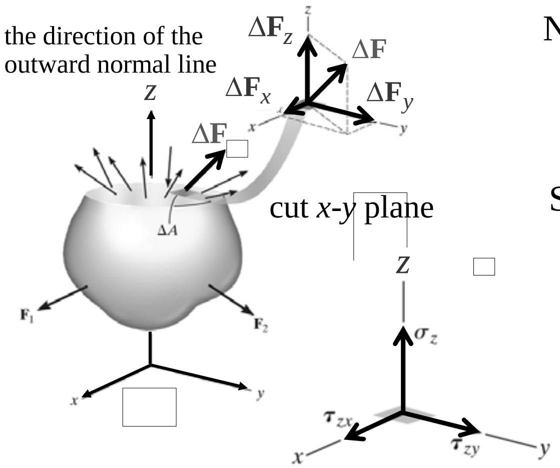


1.3 Stress

- ○ Material
  - ➢ continuous
  - ➢ cohesive



the direction of the outward normal line



Normal stress

$$\boxed{\sigma_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}}$$

Shear stress

$$\boxed{\tau_{zx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}}$$

$$\boxed{\tau_{zy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}}$$

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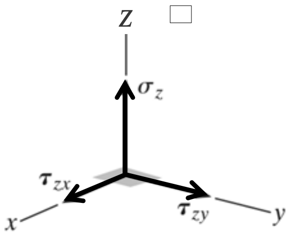
the direction of the outward normal line



x-*y* plane

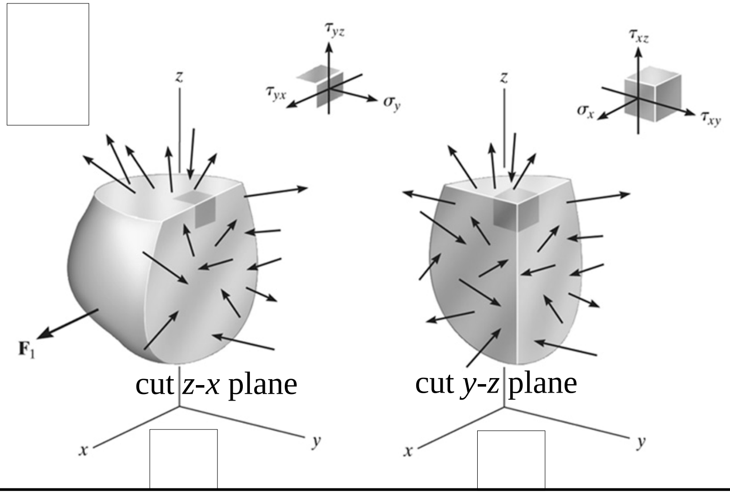
Normal stress σ_z

- pull – tensile stress
- push – compressive stress

Shear stress τ_{zx}, τ_{zy}

- *z* – the orientation of the area
- *x*, *y* – the direction lines for the shear stresses

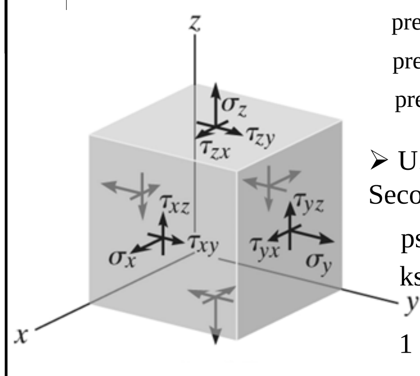
General State of Stress



cut *z*-*x* plane　　cut *y*-*z* plane

State of stress

- acting around the chosen point in the body



Units of stress

- SI system (International Standard)

  N/m^2 (pascal) (1 Pa = 1 N/m^2)

  prefix kilo- $\Rightarrow$ k ($=10^3$) kPa

  prefix mega- $\Rightarrow$ M ($=10^6$) Mpa

  prefix giga- $\Rightarrow$ G ($=10^9$) Gpa

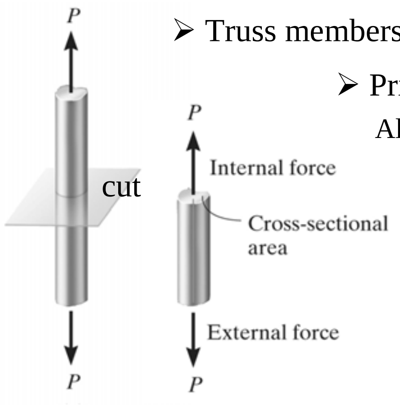
- U.S. Customary (Foot-Pound-Second)

  psi (pounds per square inch)

  ksi (kilopounds per square inch)

  1 kilopound (kip) $=$ 1000 lb.

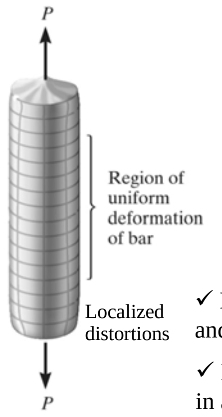
1.4 Average Normal Stress in an Axially Loaded Bar



- Truss members, hangers, and bolts
  - Prismatic
    All cross sections are the same.
  - Neglect the weight of the bar

Internal force P

= External force P



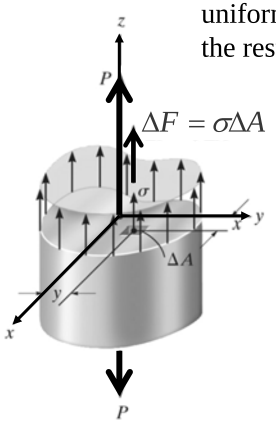
○ Assumptions

- The bar remains straight both before and after the load is applied. Also, the cross section should remain flat or plane during deformation. (deform uniformly)
- P is applied along the centroidal axis of the cross section, and the material must be homogeneous and isotropic.

✓ Homogeneous material: having the same physical and mechanical properties throughout its volume

✓ Isotropic material: having these same properties in all directions.

○ Average Normal Stress Distribution



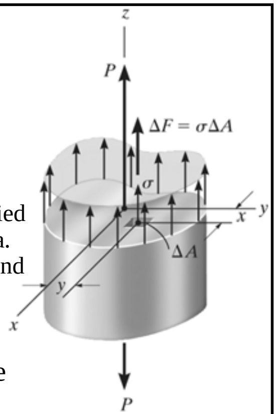
- Provided the bar is subjected to a constant uniform deformation, then this deformation is the result of a constant normal stress σ .
- Let $\Delta A \to dA$ and therefore $\Delta F \to dF$, then σ is constant.

$$+\uparrow \; F_{Rz} = \Sigma F_z$$

$$\int dF = \int_A \sigma dA$$

$$P = \sigma A \quad \Rightarrow \quad \boxed{\sigma = \frac{P}{A}}$$

$$\boxed{\sigma = \frac{P}{A}}$$



σ : average normal stress at any point on the cross-sectional area

P : internal resultant normal force, which is applied through the *centroid* of the cross-sectional area. P is determined using the method of sections and the equations of equilibrium

A : cross-sectional area of the bar

- The internal load P must pass through the centroid of the cross-section.

$$(M_R)_x = \sum M_x = 0 \; ; \quad 0 = \int_A y dF = \int_A y\sigma dA = \sigma \int_A y dA \qquad \because \int y dA = 0$$

$$(M_R)_y = \sum M_y = 0 \; ; \quad 0 = -\int_A x dF = \int_A x\sigma dA = -\sigma \int_A x dA \qquad \because \int x dA = 0$$

○ Equilibrium

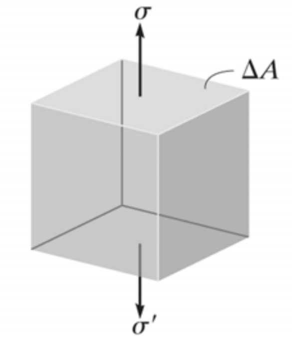
If consider vertical equilibrium of the element, then applying the equation of force equilibrium

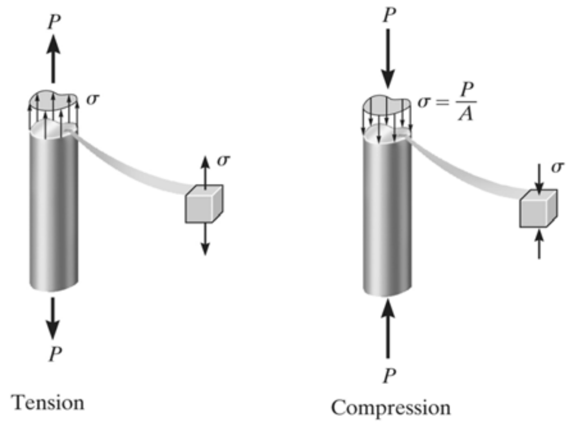
$$\sum F_z = 0$$

$$\sigma(\Delta A) - \sigma'(\Delta A) = 0$$

$$\sigma = \sigma'$$

Uniaxial stress





$P = \sigma A$ → This resultant passes through the centroid of this volume.

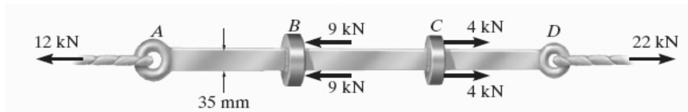
For a taped bar of rectangular cross section, for which the angle between two adjacent sides is 15°, the average normal stress, as calculated by $\sigma = P / A$, is only 2.2% less than its value found from the theory of elasticity. Therefore, this assumption can be relaxed somewhat to include bars that have a *slight taper*.

○ Maximum Average Normal Stress

- If normal stress within a bar is different from one section to the next, and if the maximum average normal stress is to be determined, then it becomes important to find the location where the ratio P/A is a maximum.
- **Axial or normal force diagram**: a plot of the normal force P versus its position x along the bar's length.
- Tension $\Rightarrow$ positive (+); Compression $\Rightarrow$ negative (-)

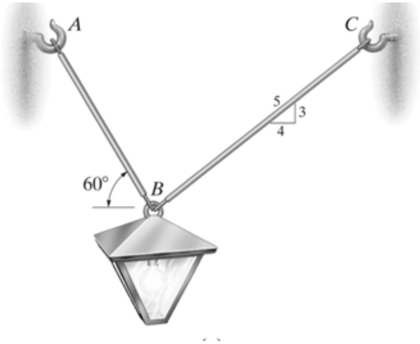
EXAMPLE 1.6

The bar in Figure has a constant width of 35 mm and a thickness of 10 mm. Determine the maximum average normal stress in the bar when it is subjected to the loading shown.



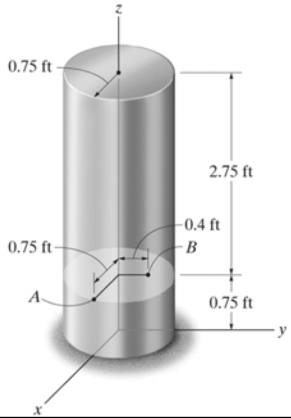
EXAMPLE 1.7

The 80-kg lamp is supported by two rods *AB* and *BC* as shown in Figure. If *AB* has a diameter of 10 mm and *BC* has a diameter of 8 mm, determine the average normal stress in each rod.



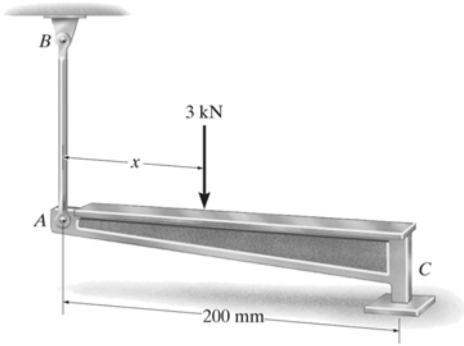
EXAMPLE 1.8

The casting shown in Figure is made of steel having a specific weight of γ_{st} = 490 lb/ft^3. Determine the average compressive stress acting at points *A* and *B*.

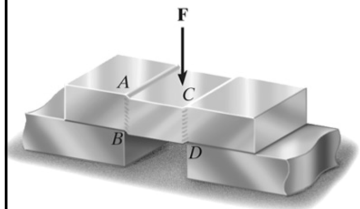


EXAMPLE 1.9

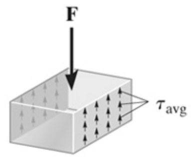
Member *AC* shown in Figure is subjected to a vertical force of 3 kN. Determine the position *x* of this force so that the average compressive stress at the smooth support *C* is equal to the average tensile stress in the tie rod *AB*. The rod has a cross-sectional area of 400 mm^2 and the contact area at *C* is 650 mm^2.



1.5 Average Shear Stress

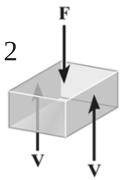


$$\boxed{\tau_{avg} = \frac{V}{A}}$$



simple or direct shear

$V = F / 2$



τ_{avg} : average shear stress at the section, which is assumed to be the same at each point located on the section

V : internal resultant shear force at the section determined from the equations of equilibrium

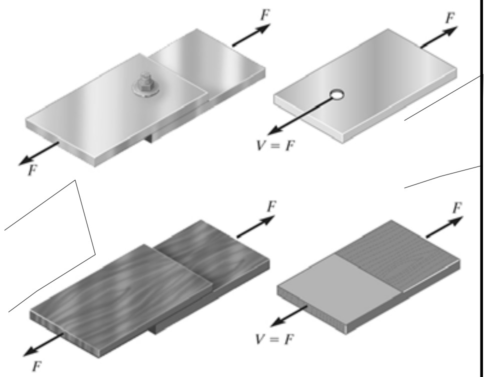
A : area at the section

✓ τ_{avg} is in the same direction as *V*.

✓ Average shear stress is only approximate.
✓ The shear-stress distribution over the critical section often reveals that much larger shear stresses occur in the material than the average shear stress.
✓ The application of average shear stress is generally acceptable for engineering design and analysis.

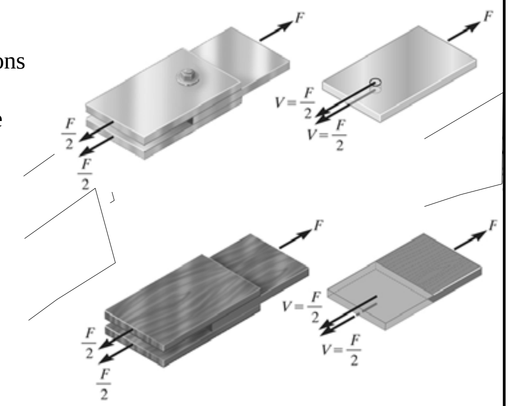
○ Single Shear

✓ single-shear connections or lap joints
✓ no friction between the members
✓ neglect the moment
✓ $V = F$

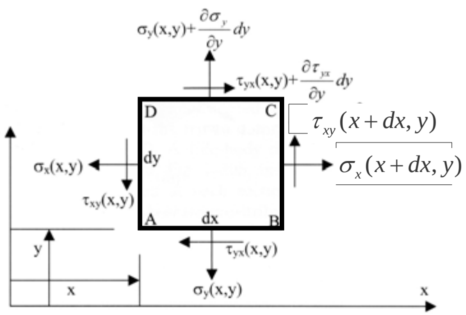


○ Double Shear

✓ double-shear connections or double lap joints
✓ no friction between the members
✓ $V = F/2$



○ Equilibrium



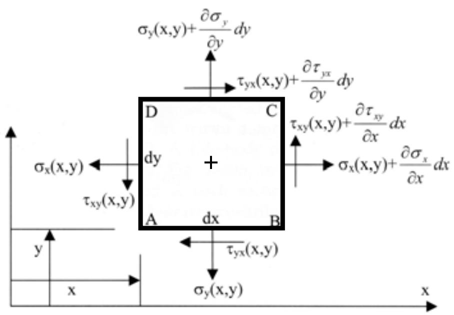
Taylor series expansion

$$\sigma_x(x+dx,y)=\sigma_x(x,y)+(\frac{\partial\sigma_x}{\partial x})dx+(\frac{\partial^2\sigma_x}{\partial^2 x})\frac{(dx)^2}{2}+\ldots$$

Neglect higher order terms

$$\sigma_x(x+dx,y)\cong\sigma_x(x,y)+(\frac{\partial\sigma_x}{\partial x})dx$$

○ Equilibrium



$\Sigma M_z = 0$ (center point)

$$\Rightarrow(\tau_{xy}+\frac{\partial\tau_{xy}}{\partial x}dx)(dy)(1)(\frac{1}{2}dx)+(\tau_{xy})(dy)(1)(\frac{1}{2}dx)-$$

$$(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}dy)(dx)(1)(\frac{1}{2}dy)-(\tau_{yx})(dx)(1)(\frac{1}{2}dy)=0$$

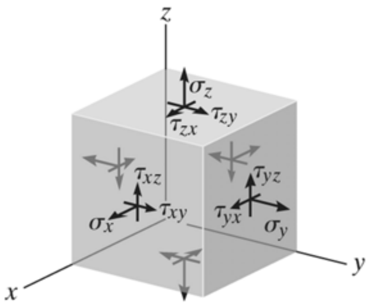
Neglect higher order terms $\Rightarrow\ \tau_{xy}=\tau_{yx}$

○ Equilibrium

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

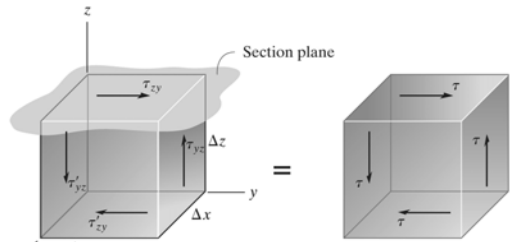
$$\tau_{zx} = \tau_{xz}$$



Six independent stress components in a general stress state

$$(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$$

○ Equilibrium



Pure shear

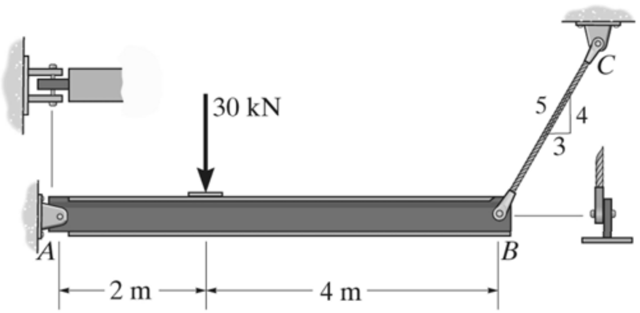
$$\Sigma M_x = 0 \Rightarrow -\tau_{zy}(\Delta x \Delta y)\Delta z + \tau_{yz}(\Delta x \Delta z)\Delta y = 0 \Rightarrow \tau_{zy} = \tau_{yz}$$

$$\Sigma F_y = 0 \Rightarrow \tau_{zy}(\Delta x \Delta y) - \tau'_{zy}(\Delta x \Delta y) = 0 \Rightarrow \tau_{zy} = \tau'_{zy}$$

$\tau_{yz} = \tau'_{yz} = \tau_{zy} = \tau'_{zy} \Rightarrow$ Complementary property

EXAMPLE 1.10

Determine the average shear stress in the 40-mm-diameter pin at *A* and the 30-mm-diameter pin at *B* that support the beam in Figure.



EXAMPLE 1.10

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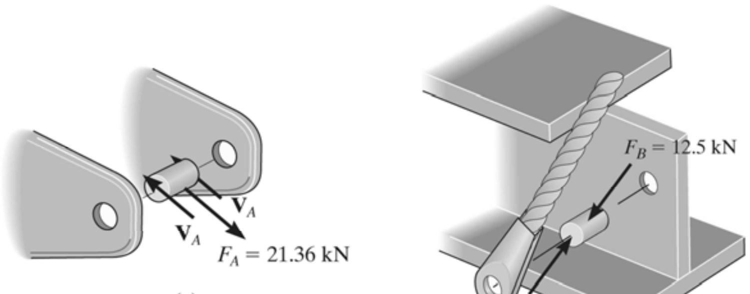
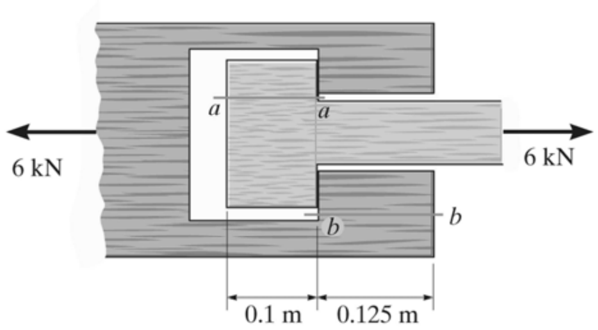


(c)

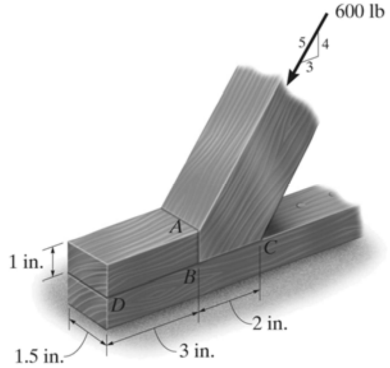
EXAMPLE 1.11

If the wood joint in Figure has a width of 150 mm, determine the average shear stress developed along shear planes *a-a* and *b-b*. For each plane, represent the state of stress on an element of the material.



EXAMPLE 1.12

The inclined member in Figure is subjected to a compressive force of 600 lb. Determine the average compressive stress along the smooth areas of contact defined by *AB* and *BC*, and the average shear stress along the horizontal plane define by *DB*.



*1.6 Allowable Stress

- ➢ An engineer in charge of the design of a structural member or mechanical element must restrict the stress in the material to a level that will be safe.
- ➢ Reasons for allowable stress
  - ✓ The load may be different from actual loadings.
  - ✓ The intended measurements of a structure or machine may not be exact, due to errors in fabrication or in the assembly of its component parts.
  - ✓ Unknown vibrations, impact, or accidental loadings can occur.
  - ✓ Atmospheric corrosion, decay, or weathering tend to cause materials to deteriorate during service.
  - ✓ Some material, such as wood, concrete, or fiber-reinforced composites, can show high variability in mechanical properties.

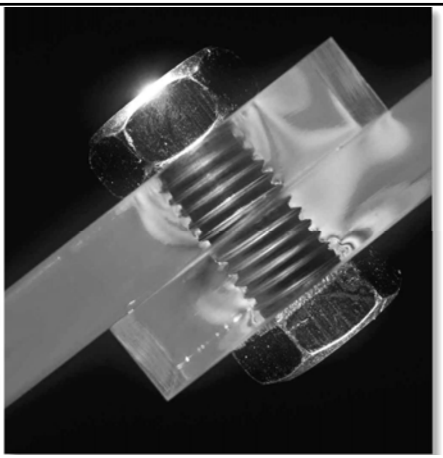
○ Factor of Safety (F.S.)

$$\boxed{F.S. = \frac{F_{\text{fail}}}{F_{\text{allow}}}}$$

F_{fail} : found from experimental testing of the material

✓ The factor of safety is selected based on experience so the uncertainties are accounted for when the member is used under similar conditions of loading and geometry.

Chapter 2
Strain

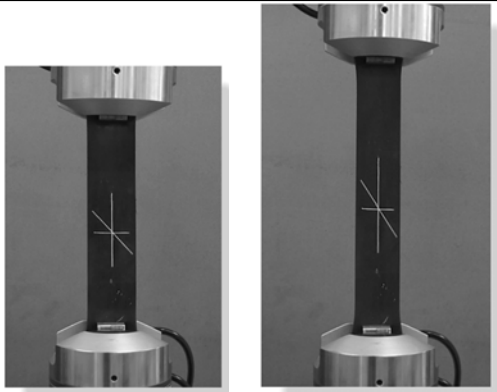


When the bolt causes compression of these two transparent plates, it produces strains in the material that shows up as a spectrum of colors when displayed under polarized light. These strains can be related to the stress in the material.

2.1 Deformation

- Deformation: A body change its shape and size when a force is applied to the body.
- Also deformation of a body can occur when the temperature of the body is changed.

Before loading



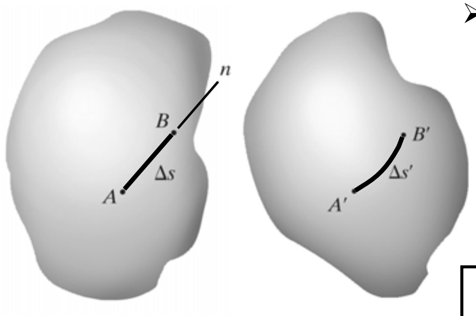
After loading

Note the before and after positions of three different line segments on this rubber membrane which is subjected to tension. The vertical line is lengthened, the horizontal line is shorten, and the inclined line changes its length and rotates.

2.2 Strain

○ Normal Strain

The elongation or contraction of a line segment per unit of length is referred to as ***normal strain***.



Undeformed body   Deformed body

- average normal strain

$$\varepsilon_{avg} = \frac{\Delta s' - \Delta s}{\Delta s}$$

- as $B \to A,\ \Delta s \to 0$

  $B' \to A',\ \Delta s' \to 0$

Normal strain at point *A* and in the direction of *n*

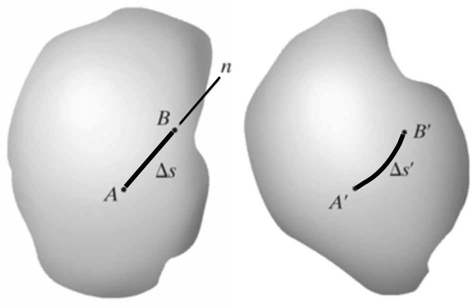
$$\boxed{\varepsilon = \lim_{B \to A \text{ along } n} \frac{\Delta s' - \Delta s}{\Delta s}}$$

Normal strain at point *A* and in the direction of *n*

$$\boxed{\varepsilon = \lim_{B \to A \text{ along } n} \frac{\Delta s' - \Delta s}{\Delta s}}$$

$$\boxed{\Delta s' \approx (1 + \varepsilon)\Delta s}$$

- $\varepsilon > 0$ – elongation
- $\varepsilon < 0$ – contraction



Undeformed body   Deformed body

Units of strain

- Normal strain is a dimensionless quantity, since it is a ratio of two lengths
- SI system (International Standard)

  meters/meter (m/m) $1\ \mu\text{m/m} = 10^{-6}\ \text{m/m}$

- U.S. Customary (Foot-Pound-Second)

  inches/inch (in./in.)

- Others

  $480(10^{-6}) = 480(10^{-6})\ \text{in./in.} = 480\ \mu\text{m/m} = 0.0480\ \%$

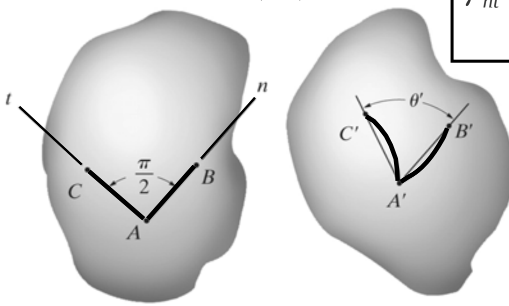
  $= 480\ \mu$

○ Shear Strain

The change in angle that occurs between two line segments that were originally *perpendicular* to one another is referred to as ***shear strain***.

- Units: radians (rad)

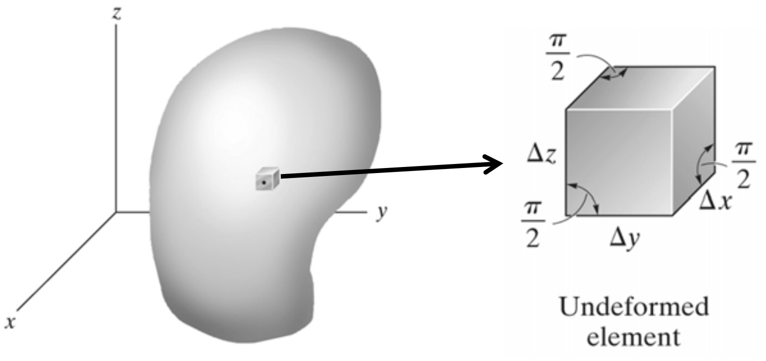
$$\boxed{\gamma_{nt} = \frac{\pi}{2} - \lim_{\substack{B \to A \text{ along } n \\ C \to A \text{ along } t}} \theta'}$$

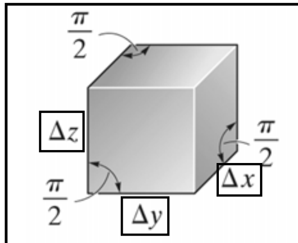


Undeformed body   Deformed body

- $\theta' < \dfrac{\pi}{2}$ – positive
- $\theta' > \dfrac{\pi}{2}$ – negative

○ Cartesian Strain Components

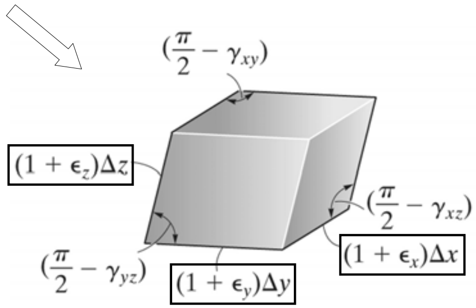




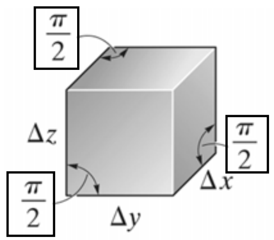
Undeformed element

✓ Normal strains cause a change in volume.

$$(1+\varepsilon_x)\Delta x \quad (1+\varepsilon_y)\Delta y \quad (1+\varepsilon_z)\Delta z$$



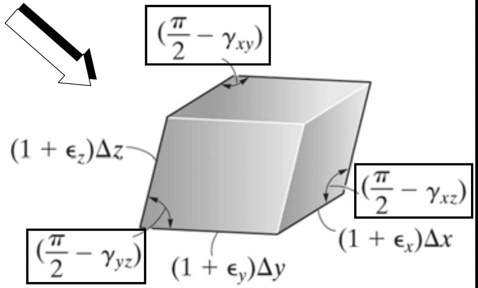
Deformed element



Undeformed element

✓ Shear strains cause a change in its shape.

$$\frac{\pi}{2}-\gamma_{xy} \quad \frac{\pi}{2}-\gamma_{yz} \quad \frac{\pi}{2}-\gamma_{xz}$$



Deformed element

➢Six components

$$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$$

○ Small Strain Analysis

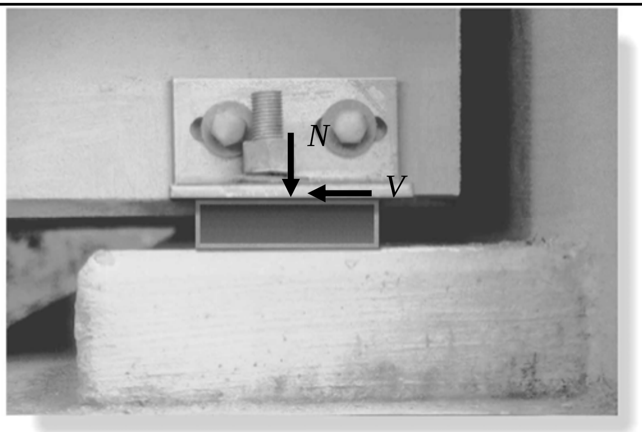
➢Most engineering design involves applications for which only *small deformations* are allowed.

normal strain $\varepsilon \ll 1$

✓The calculations for normal strain are simplified since first-order approximations can be made about their size.

➢When θ is very small, approximation is allowed as

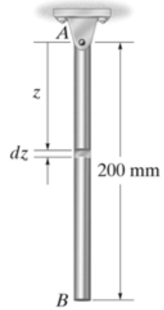
$$\sin\theta = \theta \qquad \cos\theta = 1 \qquad \tan\theta = \theta$$



The rubber bearing support under this concrete bridge girder is subjected to both normal and shear strain. The normal strain is caused by the weight and bridge loads on the girder, and the shear strain is caused by the horizontal movement of the girder due to temperature changes.

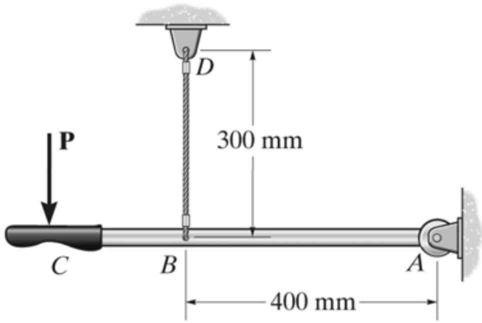
EXAMPLE 2.1

The slender rod shown in figure is subjected to an increase of temperature along its axis, which creates a normal strain in the rod of $\varepsilon_z = 40(10^{-3})z^{1/2}$, where z is given in meters. Determine (a) the displacement of the end B of the rod due to the temperature increase and (b) the average normal strain in the rod.



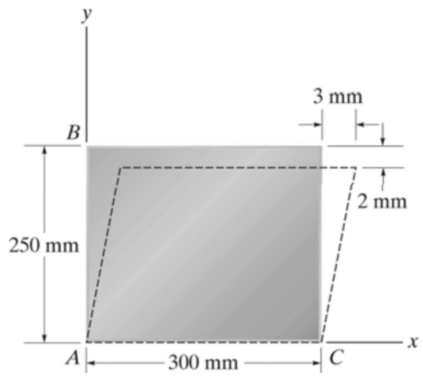
EXAMPLE 2.2

When force **P** is applied the rigid lever arm *ABC* in Fig, the arm rotates counterclockwise about pin A through an angle of $0.05°$. Determine the normal strain developed in wire *BD*.



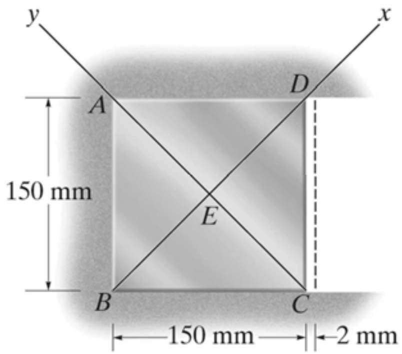
EXAMPLE 2.3

Due to a loading, the plate is deformed into the dashed shape shown in Fig. Determine (a) the average normal strain along the side *AB*, and (b) the average shear strain in the plate at *A* relative to the x and y axes.



EXAMPLE 2.4

The plate shown in Fig. is fixed connected along *AB* and held in the horizontal guides at its top and bottom, *AD* and *BC*. If its right side *CD* is given a uniform horizontal displacement of 2 mm, determine (a) the average normal strain along the diagonal *AC*, and (b) the shear strain at *E* relative to the *x*, *y* axes.



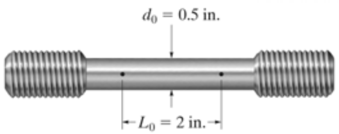
Chapter 3
Mechanical Properties of Materials

Horizontal ground displacements caused by an earthquake produced excessive strains in these bridge piers until they fractured. The material properties of the concrete and steel reinforcement must be known so that engineers can properly design this structure and thereby avoid such failures.

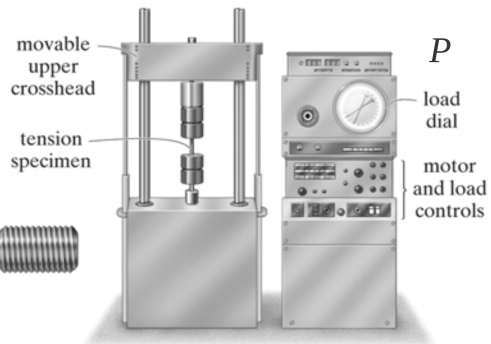
3.1 The Tension and Compression Test

○ The tension or compression test is used primarily to determine the relationship between the average normal stress and average normal strain in many engineering materials such as metals, ceramics, polymers, and composites.

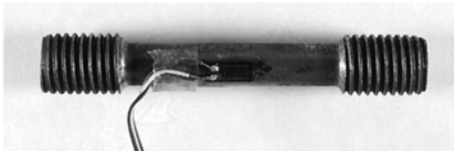
$$\delta = L - L_0$$



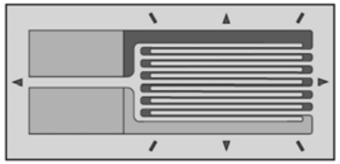
gauge-length

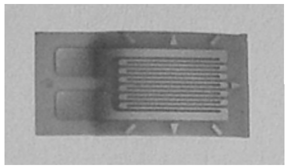


P



Typical steel specimen with attached strain gauge





electrical-resistance strain gauge

3.2 The Stress-Strain Diagram

○ Nominal or engineering stress

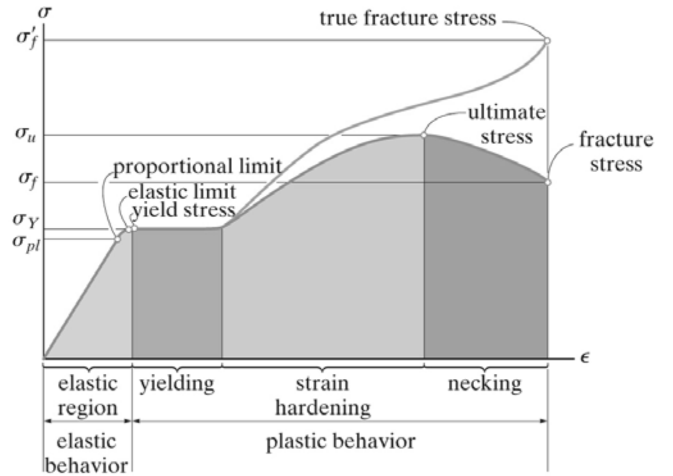
✓Divide the applied load P by the specimen's original cross-sectional area A_0
✓Stress is assumed to be constant over the cross section and throughout the region between the gauge points.

$$\sigma = \frac{P}{A_0}$$

○ Nominal or engineering strain

✓Directly from the strain gauge
✓Divide the change in the specimen's gauge length, δ , by the specimen's original gauge length L_0
✓Strain is assumed to be constant throughout the region between the gauge points.

$$\varepsilon = \frac{\delta}{L_0}$$



Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)

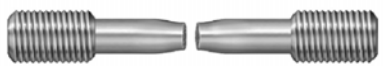
○ Elastic Behavior
- ✓ linear elastic
- ✓ proportional limit σ_{pl}
- ✓ elastic limit

○ Yielding
- ✓ yielding stress or yielding point σ_Y
- ✓ deform permanently
- ✓ plastic deformation
- ✓ perfectly plastic: continuing to elongate (strain) without any increase in load

○ Strain Hardening
- ✓ ultimate stress σ_u

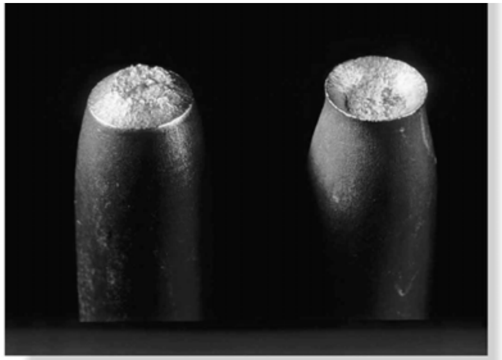
○ Necking
- ✓ fracture stress σ_f

Necking



Failure of a ductile material

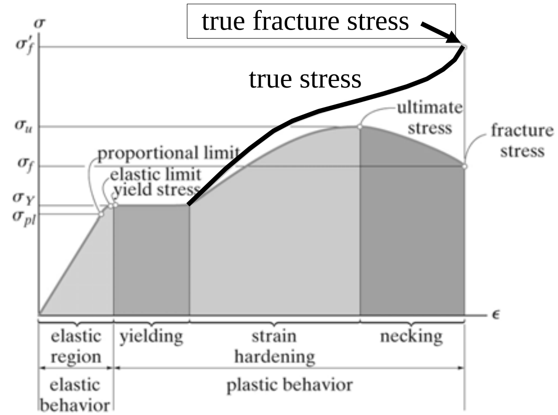
○ Necking



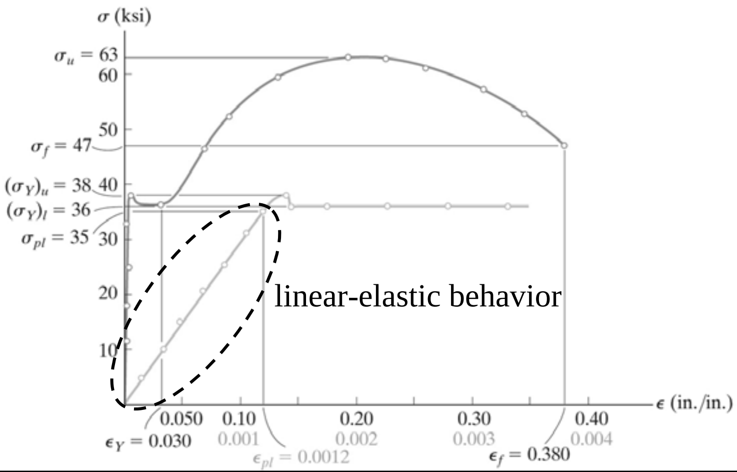
The steel specimen clearly shows the necking that occurred just before the specimen failed. This resulted in the formation of a “cup-cone” shape at the fracture location, which is characteristic of ductile materials.

○ True Stress-Strain Diagram

✓ The actual area A within the necking region is always decreasing until fracture, and so the material actually sustains increasing stress.



○ Stress-Strain Diagram for mild steel



3.3 Stress-Strain Behavior of Ductile and Brittle Materials

Materials can be classified as either being ductile or brittle, depending on their stress-strain characteristics.

○ Ductile Materials

✓A material that can be subjected to large strains before it ruptures.

✓ the ductility of a material

$$\boxed{\text{Percent elongation} = \frac{L_f - L_0}{L_0}(100\%)}$$

L_0 : the specimen’s original gauge-mark length

L_f : the fracture length

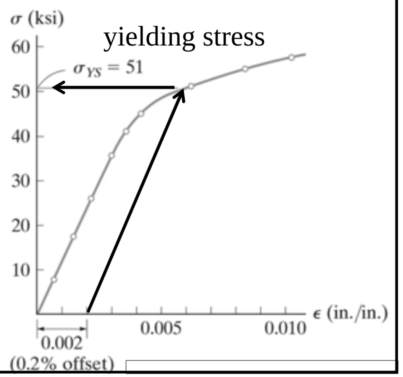
$$\text{Percent reduction of area} = \frac{A_0 - A_f}{A_0}(100\%)$$

A_0 : the specimen's original cross-sectional area

A_f : the area at fracture

✓ If a material does not have a well-define yield point, then define a yield strength using the offset method.

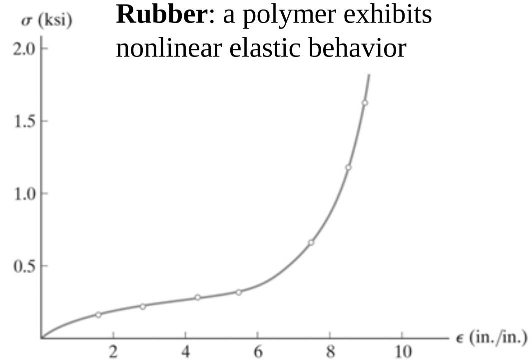
offset strain = 0.2 %



✓ In this text, we will assume that the yield strength, yield point, elastic limit, and proportional limit all coincide unless stated.

✓ Wood is a material that is often moderately ductile.

Rubber: a polymer exhibits nonlinear elastic behavior

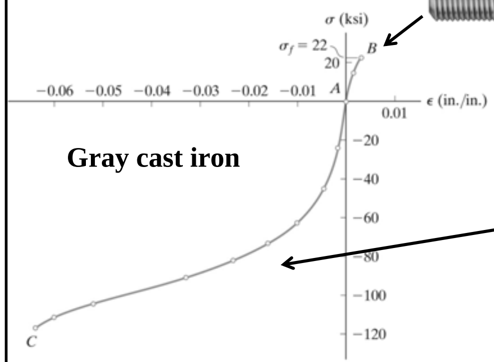


σ–ϵ diagram for natural rubber

○ Brittle Materials

✓ Materials that exhibit little or no yielding before failure are referred to as brittle materials.

Gray cast iron



σ–ϵ diagram for gray cast iron

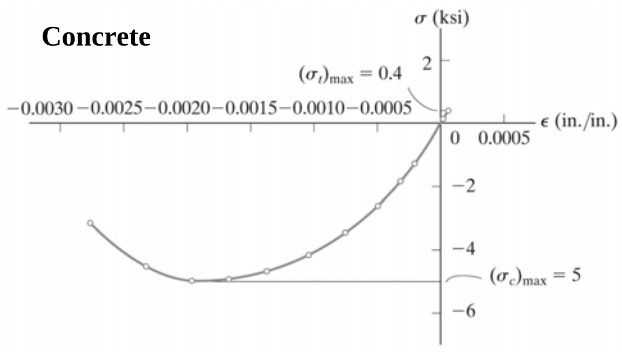

Tension failure of a brittle material


Compression causes material to bulge out

✓ The properties of concrete depend on the mix of concrete (water, sand, gravel, and cement) and the time and temperature of curing.

✓ the maximum compressive strength is almost 12.5 times greater than tensile strength.

Concrete

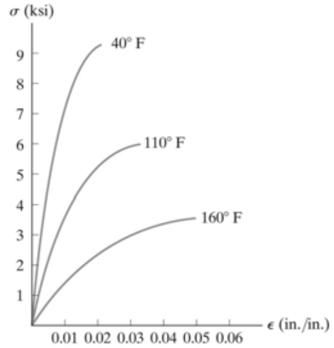


$\sigma - \epsilon$ diagram for typical concrete mix

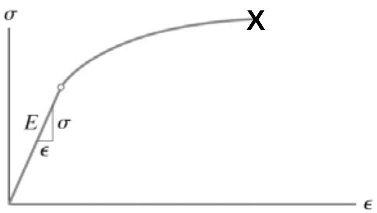
Concrete used for structural purposes must be routinely tested in compression to be sure it provides the necessary design strength for this bridge deck. The concrete cylinders shown are compression tested for ultimate stress after curing for 28 days.

✓ Most materials exhibit both ductile and brittle behavior. For example, steel has brittle behavior when it contains a high carbon content.
✓ At low temperatures materials become harder and more brittle, whereas when the temperature rises they become softer and more ductile.

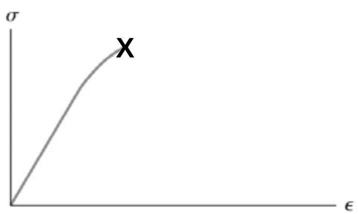
Steel rapidly loses its strength when heated.



$\sigma-\epsilon$ diagrams for a methacrylate plastic



Ductile material



Brittle material

3.4 Hooke's Law

Linear-elastic behavior

$$\boxed{\sigma = E\varepsilon}$$

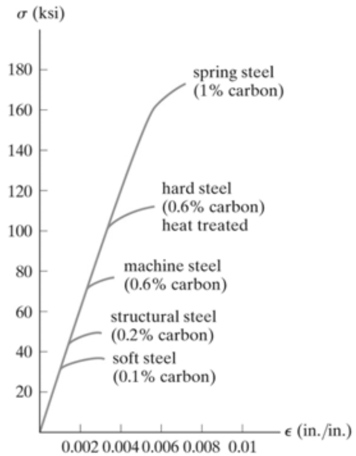
E : modulus of elasticity or Young's modulus

Units: the same as stress, e.g. psi, ksi, pascals.

For steel

$$E = \frac{\sigma_{pl}}{\varepsilon_{pl}} = \frac{35 \text{ ksi}}{0.0012 \text{ in./in.}}$$

$$= 29(10^3) \text{ ksi } = 200 \text{ GPa}$$

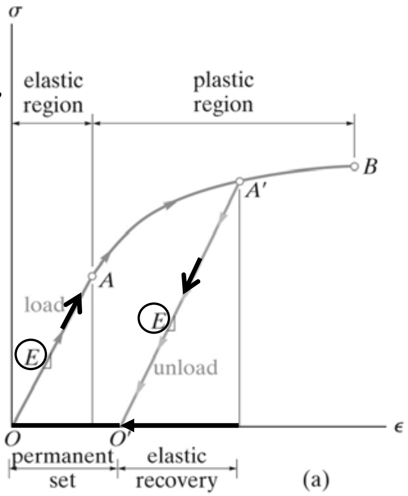


The pin was made from a hardened steel alloy, that is, one having a high carbon content. It failed due to brittle fracture.

○ Strain Hardening

If a ductile material is *loaded* into the plastic region and then *unloaded*, elastic strain is recovered as the material returns to its equilibrium state, but the plastic strain remains.

⇒ **permanent set**

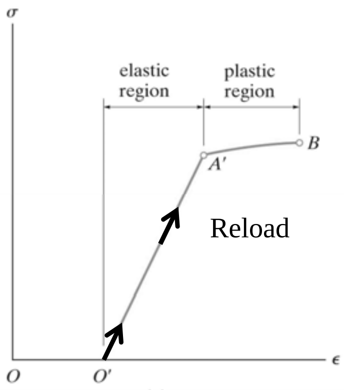


Reload

✓ yield at or near the stress *A*', higher yield point
✓ greater elastic region, less ductility

Unload and Reload

✓ in the true sense, unloading and reloading result in energy loss



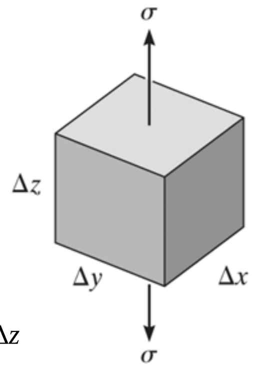
3.5 Strain Energy

As a material is deformed by an external loading, it tends to store energy internally throughout its volume. Since this energy is related to the strains in the material, it is referred to as **strain energy**.

○ **Conservation of Energy**

External Work = Internal Work

✓ The "external work" is equivalent to the "internal work" or strain energy stored in the element



Force $\Delta F = \sigma \Delta A = \sigma(\Delta x \Delta y)$

Displacement $\varepsilon\ \Delta z$

Work $\Delta W = \frac{1}{2}\Delta F\, \varepsilon \Delta z = \frac{1}{2}\sigma(\Delta x \Delta y)\varepsilon \Delta z$

Strain energy $\Delta U = \Delta W = \frac{1}{2}\sigma(\Delta x \Delta y)\varepsilon \Delta z = \frac{1}{2}\sigma\varepsilon \ \Delta V$

$$\Delta V = \Delta x \Delta y \Delta z$$

Strain-energy density

✓ the strain energy per unit volume

$$\boxed{u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\sigma\varepsilon}$$

✓ for the linear elastic material $\sigma = E\varepsilon$

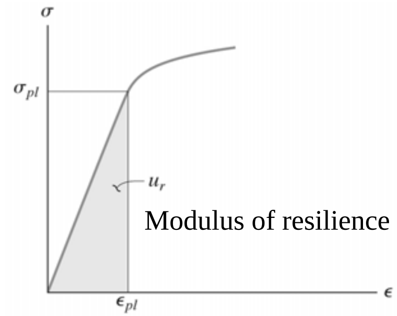
$$\boxed{u = \frac{1}{2}\frac{\sigma^2}{E}}$$

○ Modulus of Resilience

✓ When the stress reaches the proportional limit, the strain-energy is referred to as the modulus of resilience.

✓ Physically a material's resilience represents the ability of the material to absorb energy without any permanent damage to the material.

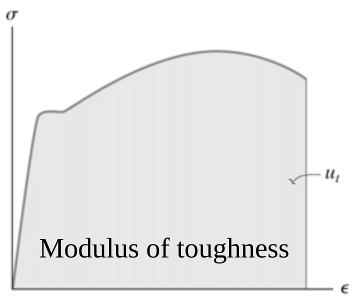
$$\boxed{u_r = \frac{1}{2}\sigma_{pl}\varepsilon_{pl} = \frac{1}{2}\frac{\sigma_{pl}^2}{E}}$$

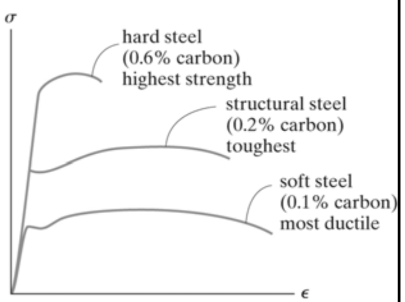


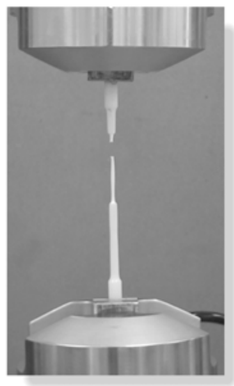
○ Modulus of Toughness

✓ Modulus of toughness represents the entire area under the stress-strain diagram.

✓ It indicates the strain-energy density of the material just before it fractures.



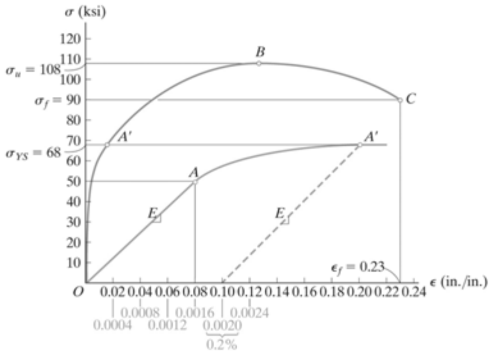




This nylon specimen exhibits a high degree of toughness as noted by the large amount of necking that has occurred just before fracture.

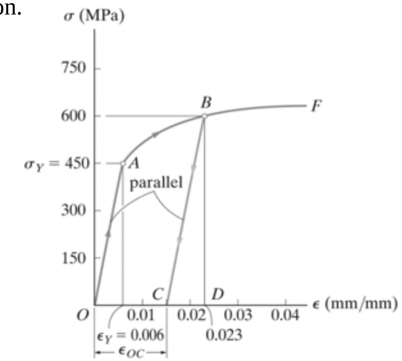
EXAMPLE 3.1

A tension test for a steel alloy results in the stress-strain diagram shown in figure. Calculate the modulus of elasticity and the yield strength based on a 0.2 % offset. Identify on the graph the ultimate stress and the fracture stress.



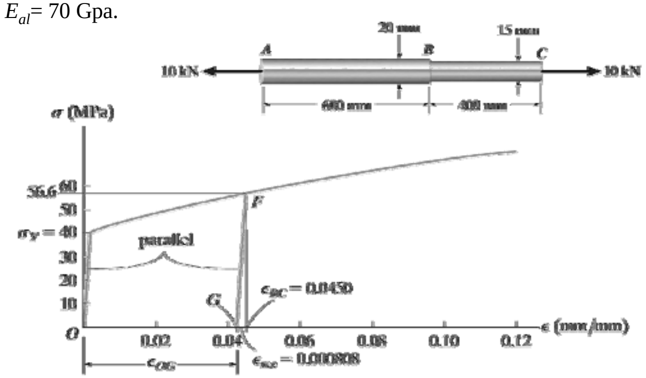
EXAMPLE 3.2

The stress-strain diagram for an aluminum alloy that is used for making aircraft parts is shown in figure. If a specimen of this material is stressed to 600 Mpa, determine the permanent strain that remains in the specimen when the load is released. Also, compute the modulus of resilience both before and after the load application.

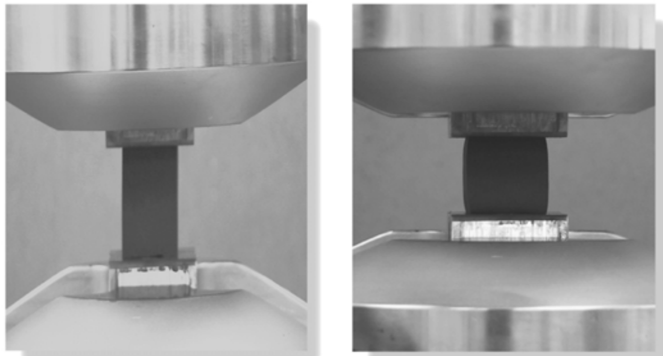


EXAMPLE 3.3

An aluminum rod shown in figure has a circular cross section and is subjected to an axial load of 10 kN. If a portion of the stress-strain diagram for the material is shown in figure, determine the approximate elongation of the rod when the load is applied. If the load is removed, what is the permanent elongation of the rod? Take E_{al}= 70 Gpa.

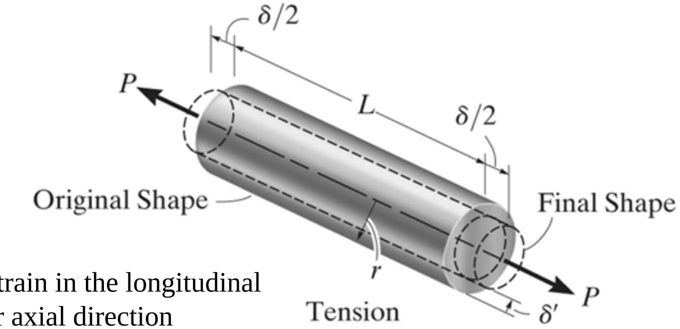


3.6 Poisson's Ratio



When the rubber block is compressed (negative strain), its sides will expand (positive strain). The ratio of these strains is constant.

When a deformable body is subjected to an axial tensile force, not only does it elongate but it also contracts laterally.

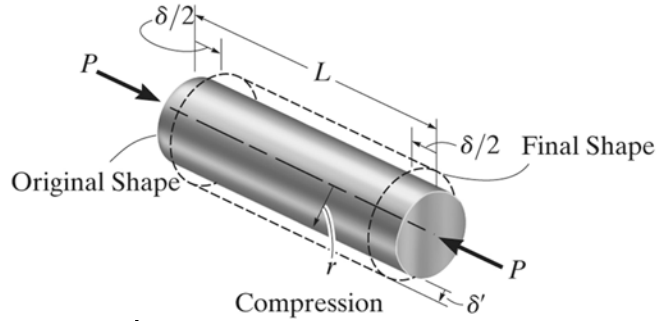


Strain in the longitudinal or axial direction

$$\boxed{\varepsilon_{\text{long}} = \frac{\delta}{L}}$$

Strain in the lateral or radial direction

$$\boxed{\varepsilon_{\text{lat}} = \frac{\delta'}{r}}$$



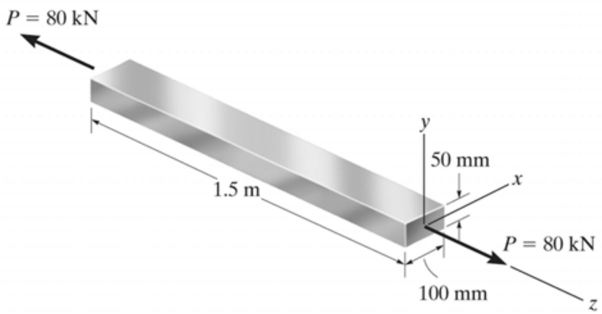
Poisson's Ratio

$$\boxed{\nu = -\frac{\varepsilon_{\text{lat}}}{\varepsilon_{\text{long}}}}$$

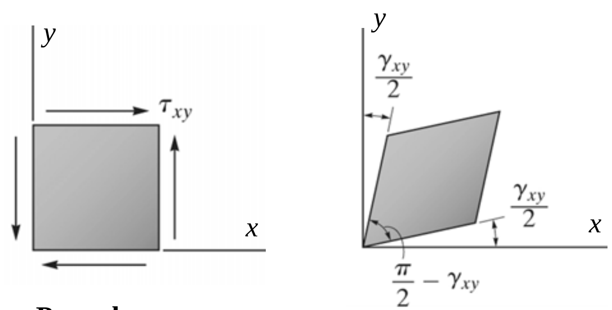
- ✓ homogeneous and isotropic material
- ✓ behave within the elastic range
- ✓ dimensionless
- ✓ for most nonporous solids $\frac{1}{4} \le \nu \le \frac{1}{3}$
- ✓ $0 \le \nu \le 0.5$

EXAMPLE 3.4

A bar made of A-36 steel has the dimensions shown in Figure. If an axial force of P = 80 kN is applied to the bar, determine the change in its length and the change in the dimensions of its cross section after applying the load. The material behaves elastically.



3.7 The Shear Stress-Strain Diagram



Pure shear

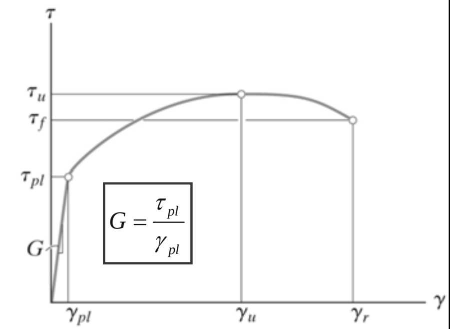
- ✓ Equilibrium requires that equal shear stresses must be developed on four faces of the element.
- ✓ The shear strain γ_{xy} measures the angular distortion of the element relative to the sides originally along the x and y axes.

Hook's law for shear

$$\boxed{\tau = G\gamma}$$

G : shear modulus of elasticity or modulus of rigidity

Units: the same as E, e.g. psi, ksi, pascals.

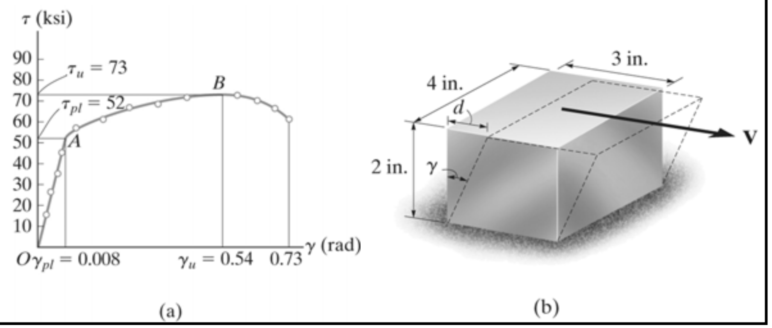


$$\boxed{G = \frac{E}{2(1+\nu)}}$$

$E_{st} = 29(10^3)$ ksi and $G_{st} = 11.0(10^3)$ ksi
so $\nu_{st} = 0.32$

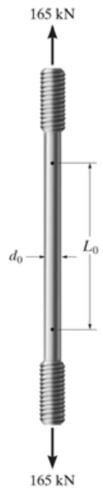
EXAMPLE 3.5

A specimen of titanium alloy is tested in torsion and the shear stress-strain diagram is shown in figure. Determine the shear modulus G, the proportional limit, and ultimate shear stress. Also, determine the maximum distance d that the top of a block of this material, shown in figure (b), could be displaced horizontally if the material behaves elastically when acted upon by a shear force **V**. What is the magnitude of **V** necessary to cause this displacement?



EXAMPLE 3.6

An aluminum specimen shown in figure has a diameter of d_0 = 25 mm and a gauge length of L_0 = 250 mm. if a force of 165 kN elongates the gauge length 1.20 mm, determine the modulus of elasticity. Also, determine by how much the force cause the diameter of the specimen to contract. Take G_{al} = 26 Gpa and σ_Y = 440 Mpa.



3.8 Failure of Materials Due to Creep and Fatigue

Load Types

- ✓ static or slowly applied load
- ✓ dynamic or fast applied load
- ✓ repeated or cycled load

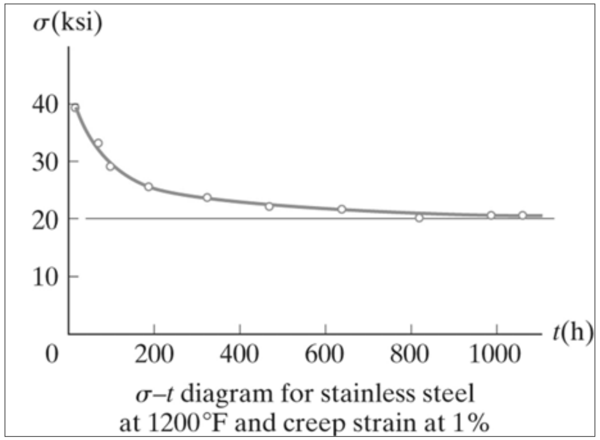
Load Environment

- ✓ at constant temperature
- ✓ at elevated temperatures

○ Creep

- ✓ When a material has to support a load for a very long period of time, it may continue to deform until a sudden fracture occurs or its usefulness is impaired.
- ✓ Time-dependent permanent deformation
- ✓ Both stress and/or temperature play a significant role in the rate of creep.
- ✓ For practical purpose, when creep becomes important, a material is usually designed to resist a specified creep strain for a given period of time.
- ✓ **Creep strength** represents the highest initial stress the material can withstand during a specified time without causing a given amount of creep strain.

The long-term application of the cable loading on this pole has caused the pole to deform due to creep.

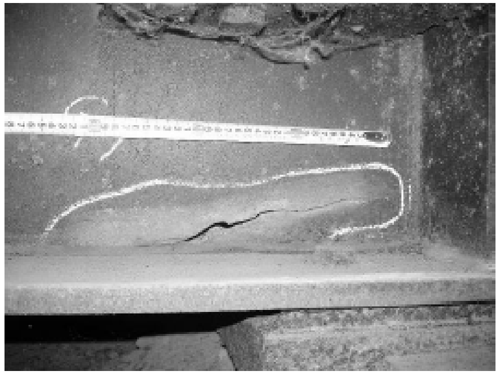


○ Fatigue

- ✓ When a material is subjected to repeated cycles of stress or strain, it causes its structure to break down, ultimately leading to fracture.
- ✓ Usually occurs in connections or supports for bridges, railroad wheels, and axles; connecting rods and crankshafts of engines; steam or gas turbine blades.
- ✓ Fracture will occur at a stress that is ***less*** than the material's yield stress.
- ✓ Since fracture occurs suddenly, the material, even though known to be ductile, behaves as if it were *brittle*.

The design of members used for amusement park rides requires careful consideration of cyclic loadings that can cause fatigue.

Engineers must account for possible fatigue failure of the moving parts of this oil-pumping rig.



Crack due to fatigue

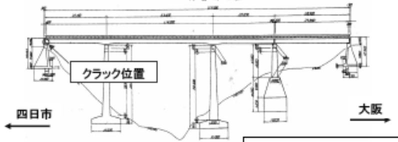
損傷事例（鋼橋）②

国土交通省

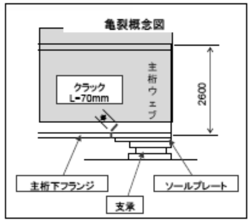
②B橋

- ・確認日時：平成18年11月22日（水）
- ・橋梁諸元：完成1972年、橋長159.01m、
  3径間連続鋼溶接非合成鈑桁橋＋単純鋼溶接非合成鈑桁橋
- ・損傷状況：桁端部支承付近の主桁下フランジ及びウェブに亀裂

［損傷位置図］



［損傷写真］



重大損傷を緊急補修した事例

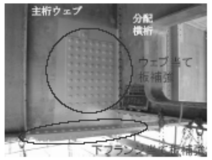
国土交通省

(2)損傷の状況とその後の対策

非合成鈑桁橋の主桁ウェブに約1m長の亀裂が発生した。亀裂発生原因は、分配横桁の主桁ウェブ貫通部から発生した疲労亀裂が冬季の低温下で脆性的に進展したと考えられる。主桁ウェブと下フランジの補強による応急復旧が終了するまで、全面通行止めを約1日実施した。

亀裂発生状況

左写真の拡大



下フランジとウェブの補強後

(3)早期に損傷を発見した場合の対応（前回の点検は平成8年度、約10年前）

亀裂長が短い段階で損傷を発見した場合、交通規制を行わずに軽微な工法（例えばストップホール等）で補修工事が実施できた可能性がある。

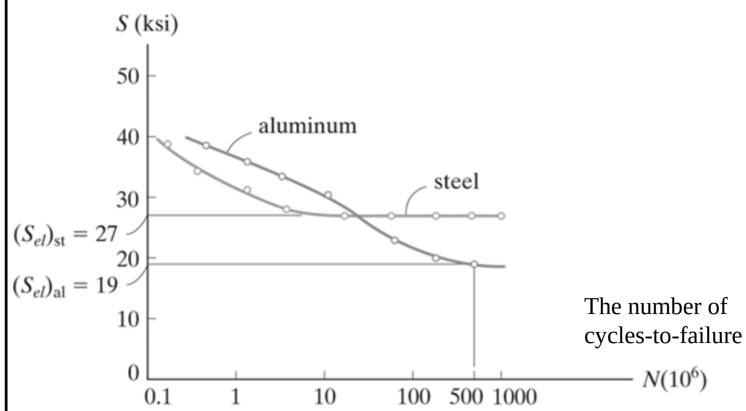
構造特性を考慮した健全度評価の重要性

◆橋梁諸元（Wikipediaより）

橋梁名： I-35W Mississippi River Bridge
形　式： Deck-arch truss bridge
架設年： 1967年
橋　長： 581m（中央径間140 m）
幅　員： 33 m（片側4車線×2 = 計8車線）
交通量： 約14万台／日

出典： ミネソタ州道路局HP

http://www.dot.state.mn.us/i35wbridge/photos/



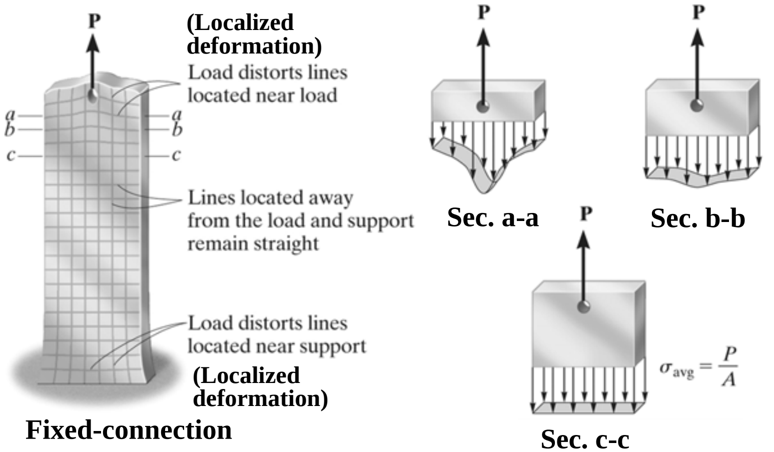
The number of cycles-to-failure N

S-N diagram or stress-cycle diagram

Chapter 4
Axial Load

The string of drill pipe suspended from this traveling block on an oil rig is subjected to extremely large loading and axial deformations.

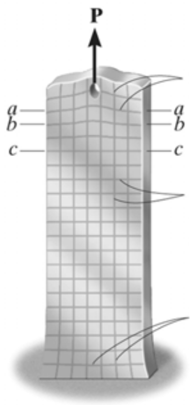
4.1 Saint-Venant's Principle

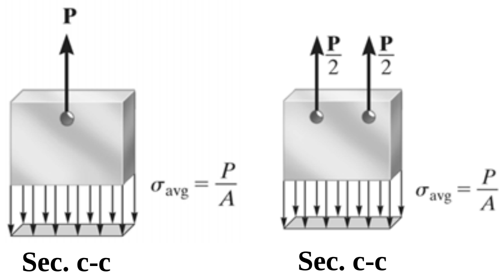


○ Saint-Venant's principle

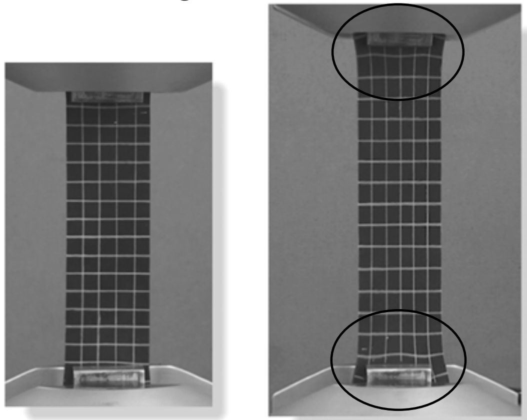
✓Both the localized deformation and stress occur within the regions of load application or at the supports tend to "even out" at a distance sufficiently removed from these regions.

✓Generally, we can consider this distance to be at least equal to the larger dimension of the loaded cross section.



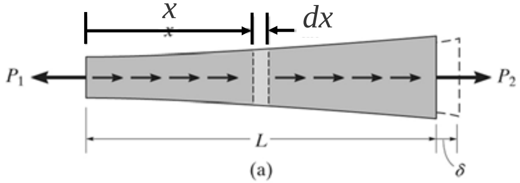


Before loading **After loading**



Notice how the lines on this rubber membrane distort after it is stretched. The localized distortions at the grips smooth out as stated by Saint-Venant's principle.

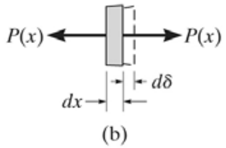
4.2 Elastic Deformation of an Axially Loaded Member



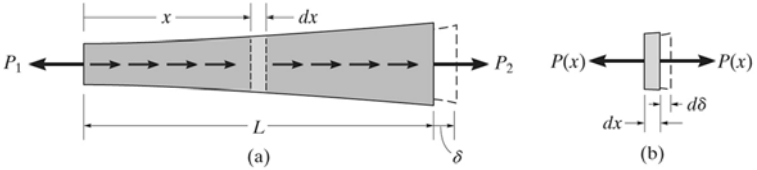
$$\sigma = \frac{P(x)}{A(x)}$$

$$\varepsilon = \frac{d\delta}{dx}$$

Linear-elastic range $\sigma = E\varepsilon$



$$\frac{P(x)}{A(x)} = E\left(\frac{d\delta}{dx}\right) \rightarrow d\delta = \frac{P(x)\,dx}{A(x)E}$$



$$d\delta = \frac{P(x)\,dx}{A(x)E} \rightarrow \boxed{\delta = \int_0^L \frac{P(x)\,dx}{A(x)E}}$$

δ : displacement of one point on the bar relative to another point

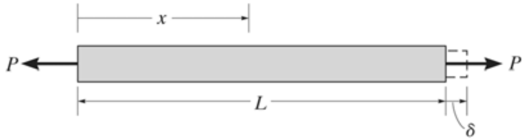
L : original distance between the points

$P(x)$: internal axial force at the section, located a distance x from one end

$A(x)$: cross-sectional area of the bar, expressed as a function of x

E : modulus of elasticity for the material

○ Constant Load and Cross-Sectional Area



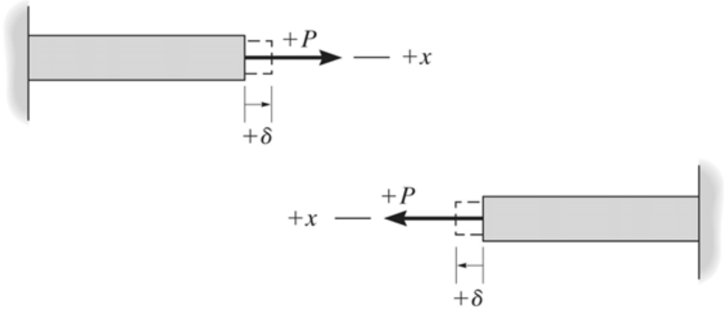
Relative displacement = Deformation

$$\delta = \int_0^L \frac{P(x)\,dx}{A(x)E} \rightarrow \boxed{\delta = \frac{PL}{AE}}$$

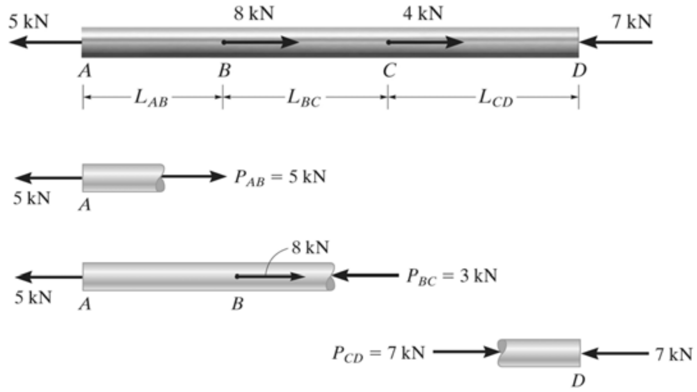
If the bar is subjected to several different axial forces, or the cross-sectional area or modulus of elasticity changes abruptly from one region of the bar to the next,

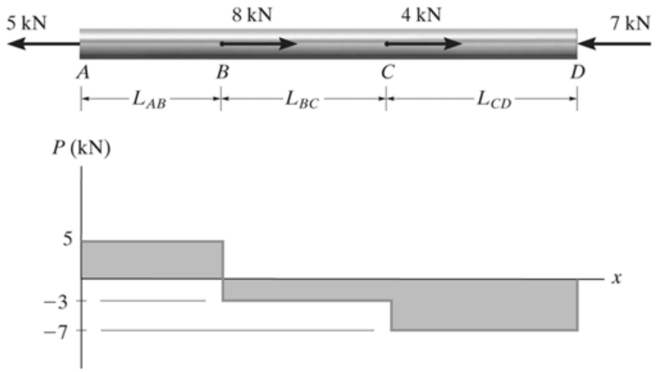
$$\rightarrow \boxed{\delta = \sum \frac{PL}{AE}}$$

○ **Sign Convention**



- Positive (+): Tension and elongation
- Negative(–): Compression and contraction



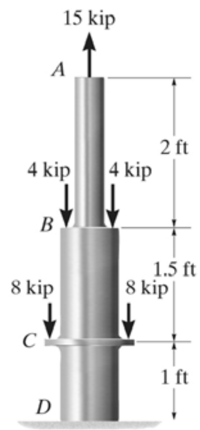


Relative displacement

$$\delta_{A/D} = \sum \frac{PL}{AE} = \frac{(5\text{ kN})L_{AB}}{AE} + \frac{(-3\text{ kN})L_{BC}}{AE} + \frac{(-7\text{ kN})L_{CD}}{AE}$$

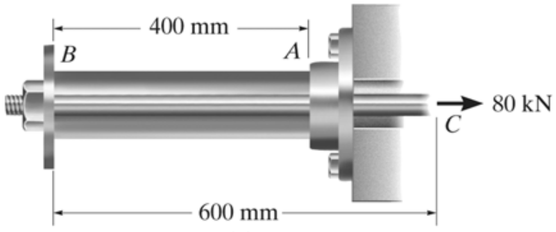
EXAMPLE 4.1

The composite A-36 steel bar shown in figure is made from two segments, *AB* and *BD*, having cross-sectional area of A_{AB} = 1 in^2 and A_{BD} = 2 in^2. Determine the vertical displacement of end *A* and the displacement of *B* relative to *C*.



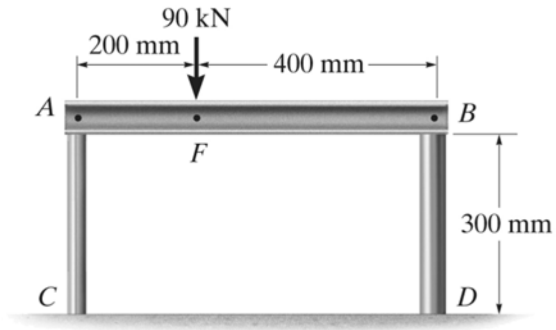
EXAMPLE 4.2

The assembly shown in figure consists of an aluminum tube *AB* having a cross-sectional area of 400 mm^2. A steel rod having a diameter of 10 mm is attached to a rigid collar and passes through the tube. If a tensile load of 80 kN is applied to the rod, determine the displacement of the end *C* of the rod. Take E_{st} = 200 GPa, E_{al} = 70 Gpa.



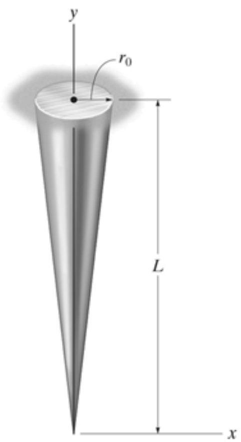
EXAMPLE 4.3

A rigid beam *AB* rests on the two short posts shown in figure. *AC* is made of steel and has a diameter of 20 mm, and *BD* is made of aluminum and has a diameter of 40 mm. Determine the displacement of point *F* on *AB* if a vertical load of 90 kN is applied over this point. Take E_{st} = 200 GPa, E_{al} = 70 GPa.



EXAMPLE 4.4

A member is made from a material that has a specific weight γ and modulus of elasticity *E*. if it is formed into a cone having the dimensions shown in figure, determine how far its end is displaced due to gravity when it is suspended in the vertical position.

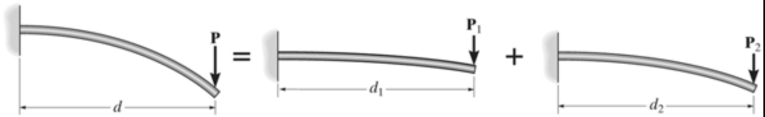


4.3 Principle of Superposition

- The loading must be linearly related to the stress or displacement that is to be determined.

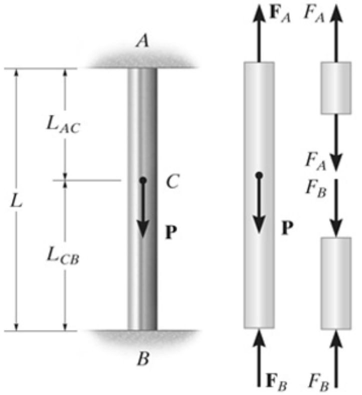
$$\sigma = \frac{P}{A} \Rightarrow \sigma \propto P \qquad \delta = \frac{PL}{AE} \Rightarrow \delta \propto P$$

- The loading must not significantly change the original geometry or configuration of the member



$$\because d_1 \neq d_2 \neq d \qquad \therefore Pd \neq P_1 d_1 + P_2 d_2$$

4.4 Statically Indeterminate Axially Loaded Member



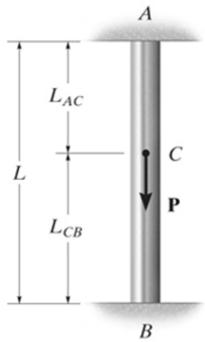
$$+\uparrow \sum F = 0$$

$$F_B + F_A - P = 0$$

Statically indeterminate

✓ The equilibrium equation(s) are not sufficient to determine the reactions.

Statically indeterminate



✓ Consider the geometry of the deformation to establish an additional equation needed for solution.
✓ An equation that specifies the conditions for displacement is referred to as a ***compatibility*** or ***kinematic condition***.

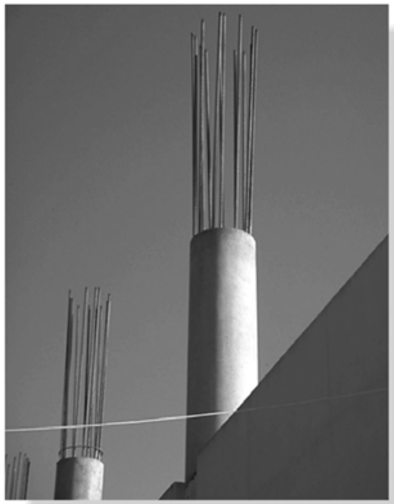
Compatibility equation $\delta_{A/B} = 0$

$$\frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} = 0 \qquad \textbf{(1)}$$

Equilibrium equation

$$F_B + F_A - P = 0 \qquad \textbf{(2)}$$

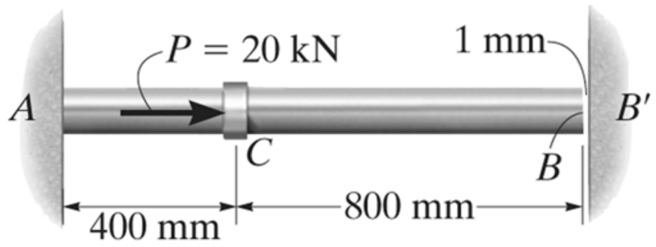
Solve (1) and (2) ➔ $F_A = P\left(\frac{L_{CB}}{L}\right)$ and $F_B = P\left(\frac{L_{AC}}{L}\right)$

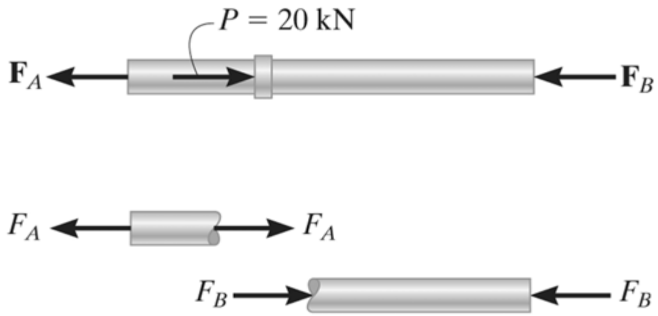


Most concrete columns are reinforced with steel rods; and since these two materials work together in supporting the applied load, the forces in each material become statically indeterminate.

EXAMPLE 4.5

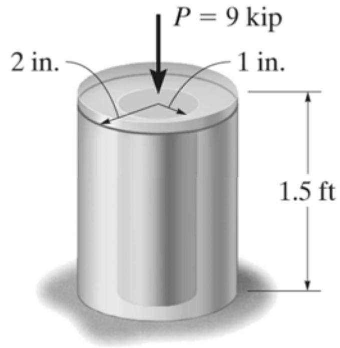
The steel rod shown in figure has a diameter of 5 mm. It is attached to the fixed wall at *A*, and before it is loaded, there is a gap between the wall at B' and the rod of 1 mm. Determine the reactions at *A* and B' if the rod is subjected to an axial force of P = 20 kN as shown. Neglect the size of the collar at *C*. Take E_{st} = 200 GPa

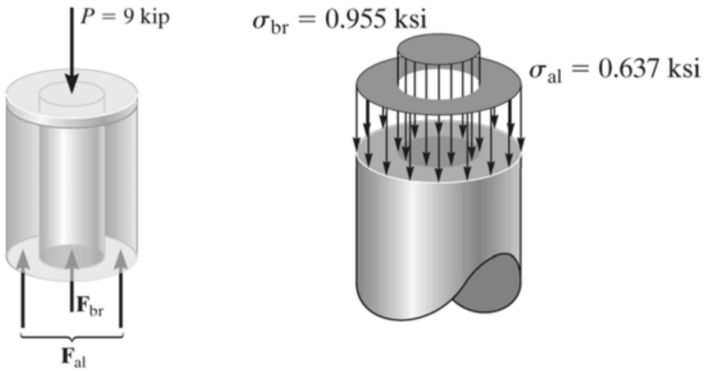




EXAMPLE 4.6

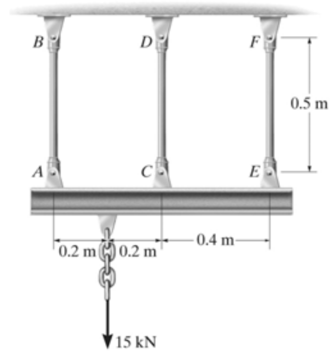
The aluminum post shown in figure is reinforced with a brass core. If this assembly supports a resultant axial compressive load of P = 9 kip, applied to the rigid cap, determine the average normal stress in the aluminum and the brass. Take $E_{al} = 10(10)^3$ ksi and $E_{br} = 15(10)^3$ ksi

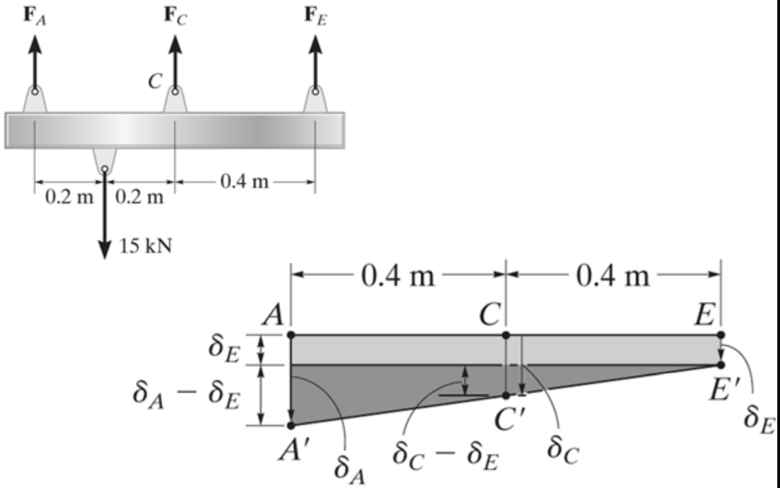




EXAMPLE 4.7

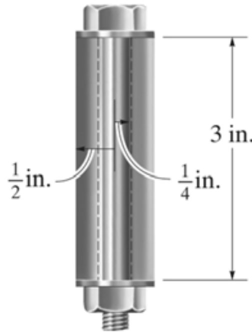
The three A-36 steel bars shown in figure are pin connected to a rigid member. If the applied load on the member is 15 kN, determine the force developed in each bar. Bars AB and EF each have a cross-sectional area of 25 mm^2, and bar CD has a cross-sectional area of 15 mm^2.

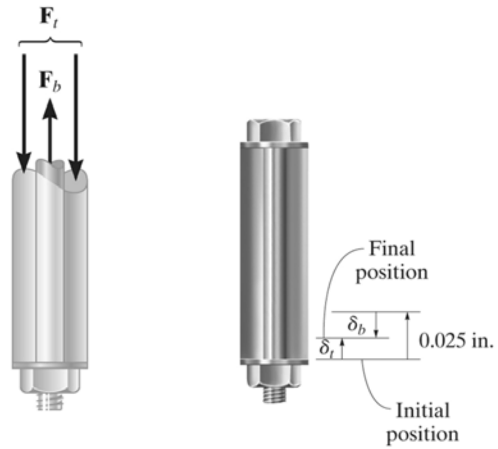




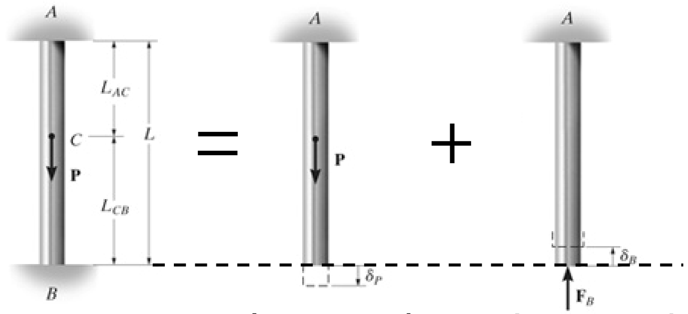
EXAMPLE 4.8

The bolt shown in figure is made of 2014-T6 aluminum alloy and is tightened so it compresses a cylindrical tube made of Am 1004-T61 magnesium alloy. The tube has an outer radius of ½ in., and it is assumed that both the inner radius of the tube and the radius of the bolt are ¼ in. The washers at the top and bottom of the tube are considered to be rigid and have a negligible thickness. Initially the nut is hand-tightened slightly; then, using a wrench, the nut is further tightened one-half turn. If the bolt has 20 threads per inch, determine the stress in the bolt.





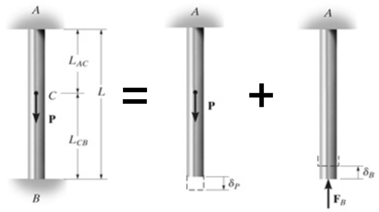
4.5 The Force Method of Analysis for Axially Loaded Members



No displacement at B

Displacement at B when redundant force at B is removed

Displacement at B when only the redundant force at B is applied



Compatibility equation

$$(+\downarrow) \qquad 0 = \delta_p - \delta_B$$

Apply the load-displacement relationship $\delta = PL/AE$

$$\delta_p = PL_{AC}/AE \qquad \delta_B = F_B L/AE$$

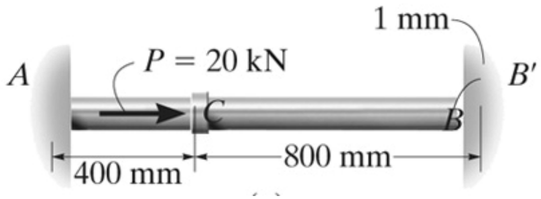
$$\rightarrow \quad 0 = \frac{PL_{AC}}{AE} - \frac{F_B L}{AE} \quad \Rightarrow \quad \boxed{F_B = P\left(\frac{L_{AC}}{L}\right)}$$

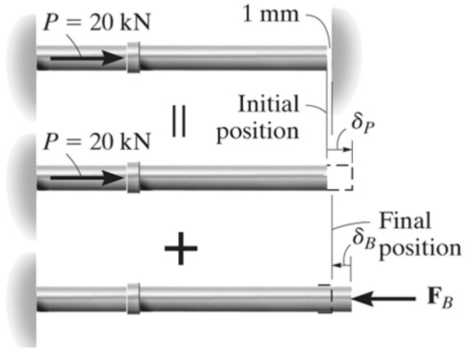
From the equation of equilibrium

$$+\uparrow \sum F = 0 \qquad P\left(\frac{L_{AC}}{L}\right) + F_A - P = 0 \qquad L_{CB} = L - L_{AC} \quad \Rightarrow \quad \boxed{F_A = P\left(\frac{L_{CB}}{L}\right)}$$

EXAMPLE 4.9

The A-36 steel rod shown in figure has a diameter of 5 mm. It is attached to the fixed wall at A, and before it is loaded there is a gap between the wall at B' and the rod of 1 mm. Determine the reactions at A and B'.





Compatibility equation $(+\rightarrow) \quad 0.001 = \delta_p - \delta_B$

4.6 Thermal Stress

Most traffic bridges are designed with expansion joints to accommodate the thermal movement of the deck and thus avoid any thermal stress.

Long extensions of ducts and pipes that carry fluids are subjected to variations in climate that will cause them to expand and contract. Expansion joints, such as the one shown, are used to mitigate thermal stress in the material.

○ A change in temperature can cause a material to change its dimensions. If the temperature increases, generally a material expands, whereas if the temperature decrease, the material will contract.

Thermal deformation $\boxed{\delta_T = \alpha\, \Delta T\, L}$

δ_T : the algebraic change in length of the member

α : a property of the material, referred to as the linear coefficient of thermal expansion. The units measure strain per degree of temperature.

1/°F (FPS system) 1/°C (SI system)

ΔT : the algebraic change in temperature of the member

L : the original length of the member

○ If the change in temperature or/and α varies throughout the length of the member,

$$\boxed{\delta_T = \int_0^L \alpha(x)\, \Delta T(x)\, dx}$$

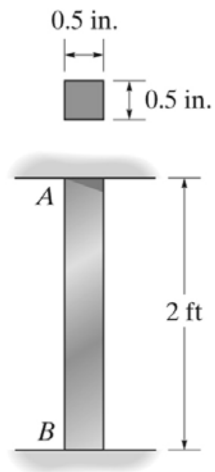
Important Remarks

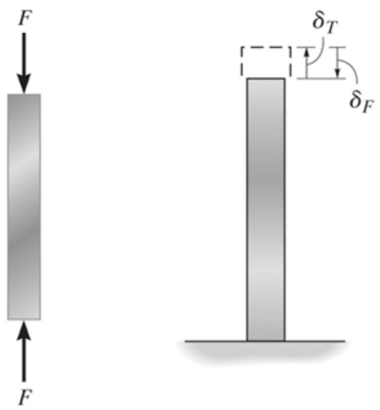
✓ The change in length of a ***statically determinate*** member can readily be computed, since the member is free to expand or contract when it undergoes a temperature change.

✓ However, in a ***statically indeterminate*** member, these thermal displacements can be constrained by the support, producing **thermal stresses** that must be considered in design.

EXAMPLE 4.10

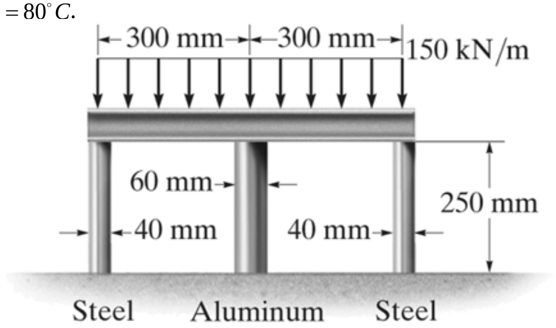
The A-36 steel bar shown in figure is constrained to just fit between two fixed supports when $T_1 = 60^\circ F$. If the temperature is raised to $T_2 = 120^\circ F$, determine the average normal thermal stress developed in the bar.

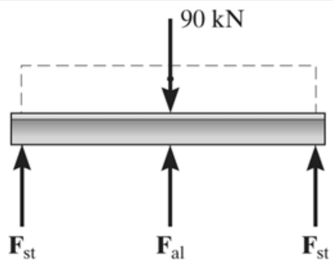


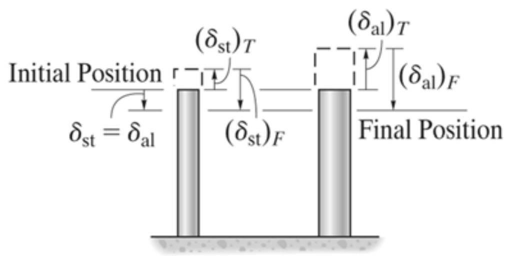


EXAMPLE 4.11

The rigid beam shown in figure is fixed to the top of the three posts made of A-36 steel and 2014-T6 aluminum. The posts each have a length of 250 mm when no load is applied to the beam, and the temperature is $T_1 = 20^\circ C$. Determine the force supported by each post if the bar is subjected to a uniform distributed load of 150 kN/m and the temperature is raised to $T_2 = 80^\circ C$.

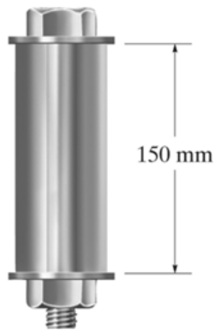


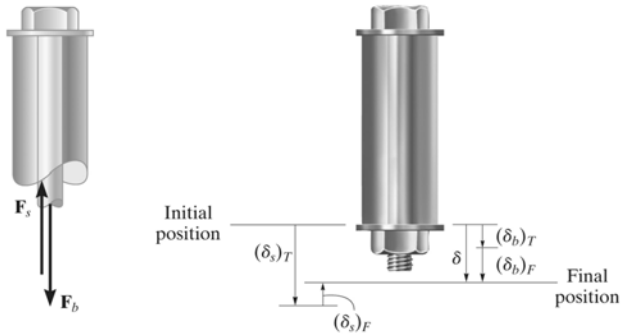




EXAMPLE 4.11

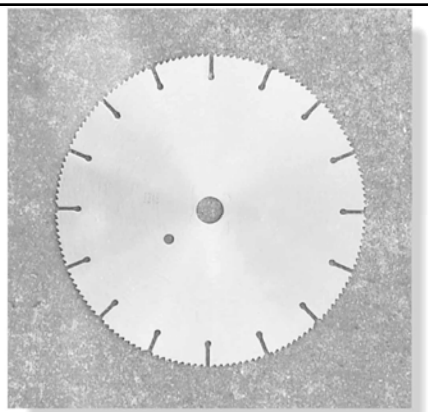
A 2014-T6 aluminum tube having a cross-section area of 600 mm^2 is used as a sleeve for an A-36 steel bolt having a cross-sectional area of 400 mm^2. When the temperature is $T_1 = 15^\circ C$, the nut holds the assembly in snug position such that the axial force in the bolt is negligible. If the temperature increases to $T_2 = 80^\circ C$, determine the average normal stress in the bolt and sleeve.



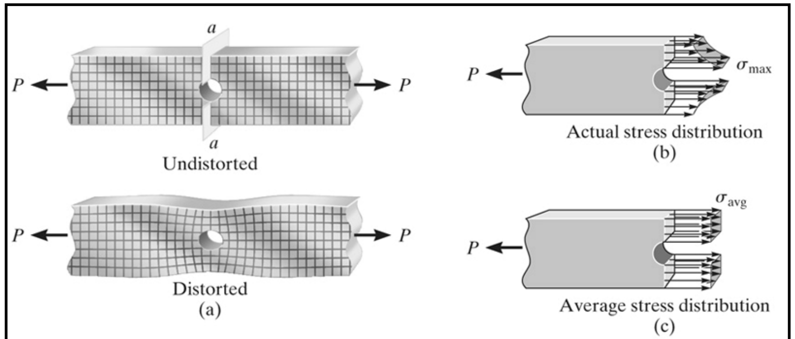


4.7 Stress Concentrations

Stress concentrations often arise at sharp corners on heavy machinery. Engineers can mitigate this effect by using stiffeners welded to the corners.



This saw blade has grooves cut into it in order to relieve both the dynamic stress that develops within it as it rotates and the thermal stress that develops as it heats up. Note the small circles at the end of each groove. These serve to reduce the stress concentrations that develop at the end of each groove.

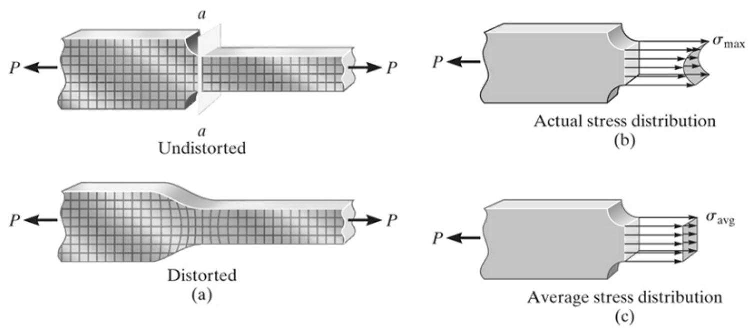


Resultant force $P = \int_A \sigma \, dA$

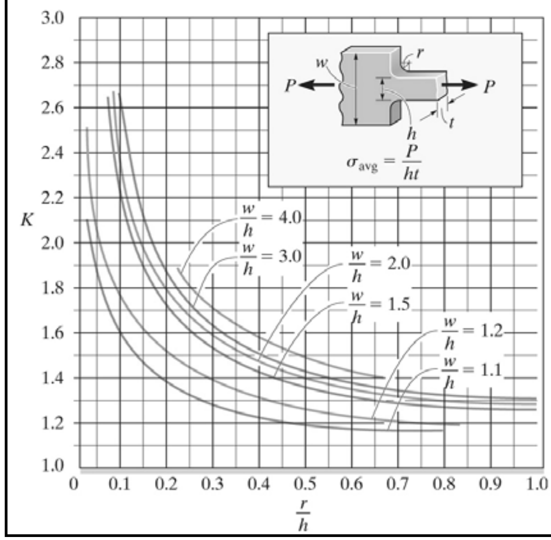
Average stress $\sigma_{avg} = P / A$

Stress-concentration factor K

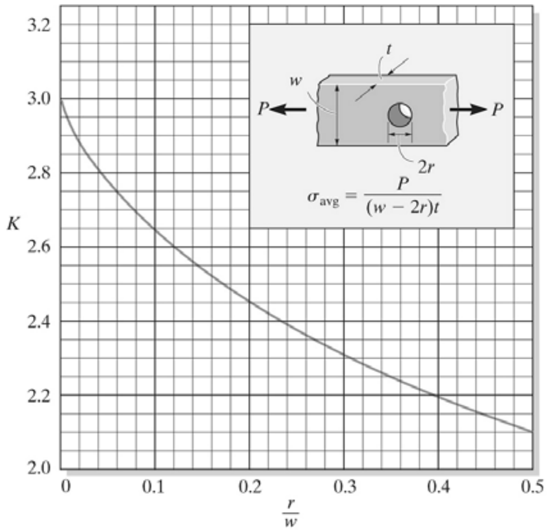
$$\boxed{K = \frac{\sigma_{max}}{\sigma_{avg}}}$$

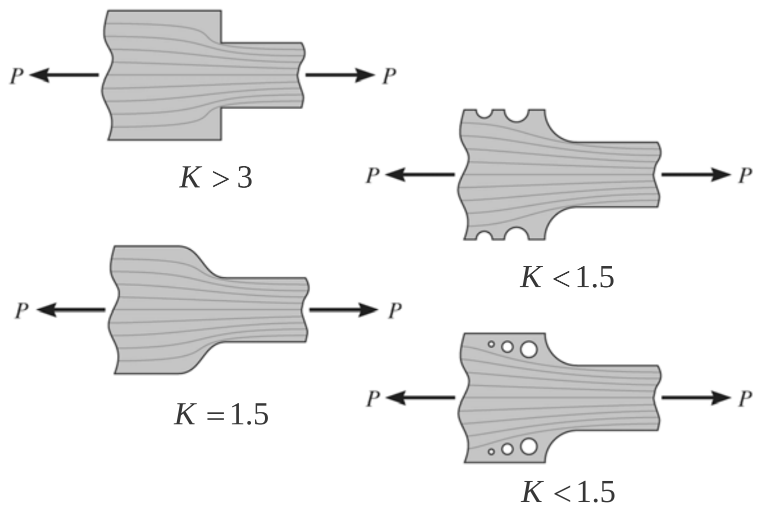


$$\sigma_{max} = K\sigma_{avg}$$



✓ Stress-concentration factor K is independent of the bar's material properties; rather, it depends only on the bar's geometry and the type of discontinuity.
✓ As the size r of the discontinuity is decreased, the stress-concentration is increased.





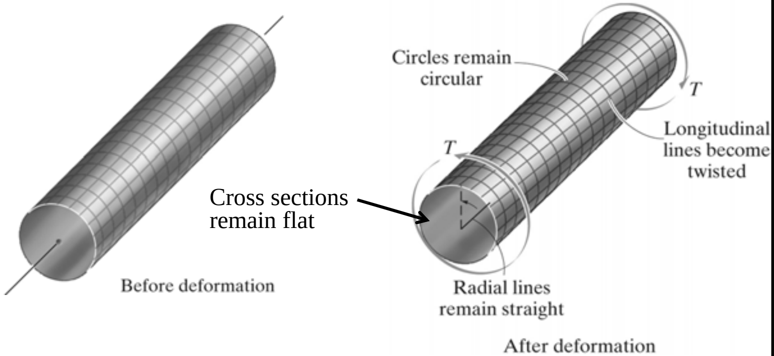
✓Cracking of the concrete has occurred at all the corners of this slab due to shrinkage of the concrete while it cured. These stress concentrations can be avoided by making the hole circular.
✓ Stress concentrations are also responsible for many failures of structural members or mechanical elements subjected to fatigue loadings.
✓ The material localized at the tip of the crack remains in a brittle state, and so the crack continues to grow, leading to a progressive fracture.

Failure of this steel pipe in tension occurred at its smallest cross-sectional area, which is through the hole. Notice how the material yielded around the fractured surface.

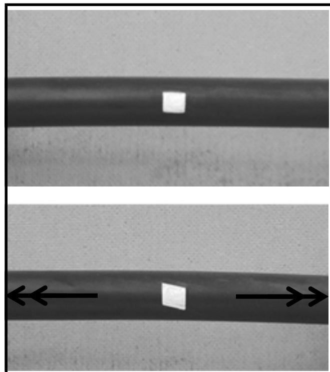
Chapter 5
Torsion

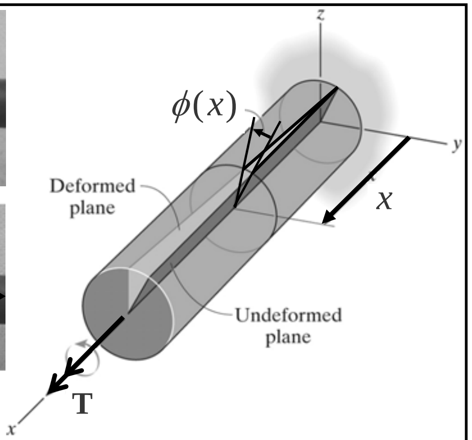
The torsional stress and angle of twist of this soil auger depend upon the output of the machine turning the bit as well as the resistance of the soil in contact with the shaft.

5.1 Torsional Deformation of a Circular Shaft

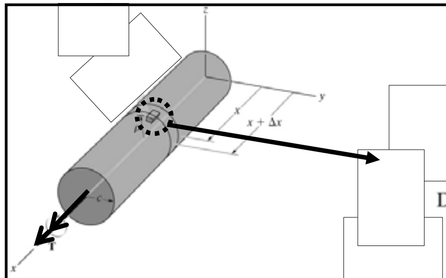


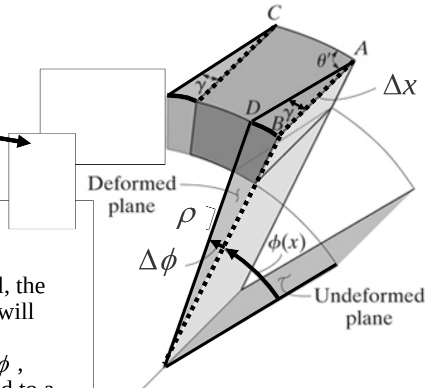
✓ If the angle of rotation is small, the length of the shaft and its radius will remain unchanged.





$\phi(x)$: angle of twist





✓ If the angle of rotation is small, the length of the shaft and its radius will remain unchanged.
✓ The difference in rotations, $\Delta\phi$, causes the element to be subjected to a shear strain.

$$BD = \rho\, d\phi = dx\, \gamma$$

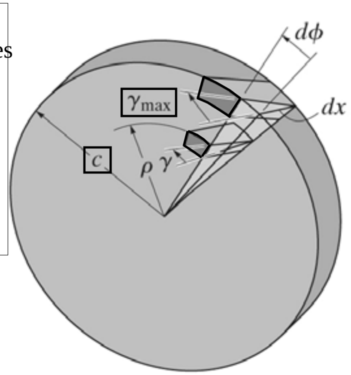
Shear strain $\gamma = \dfrac{\pi}{2} - \lim\limits_{\substack{B \to A \text{ along } CA \\ C \to A \text{ along } BA}} \theta'$

$$\gamma = \rho \frac{d\phi}{dx}$$

$$\gamma = \rho \frac{d\phi}{dx}$$

$\dfrac{d\phi}{dx}$ = constant over the cross section

✓ The shear strain within the shaft varies linearly along any radial line, from zero at the axis of the shaft to a maximum at its outer boundary.



$$\frac{d\phi}{dx} = \frac{\gamma}{\rho} = \frac{\gamma_{\max}}{c}$$

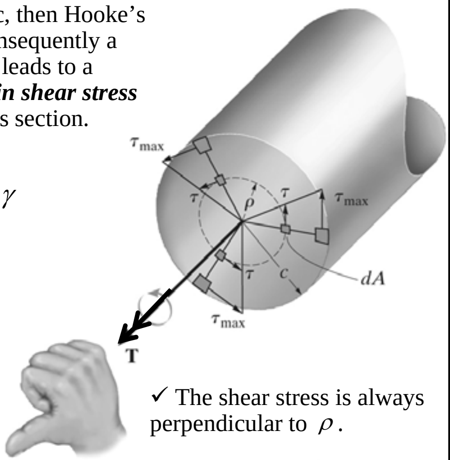
$$\Rightarrow \quad \gamma = \left(\frac{\rho}{c}\right)\gamma_{\max}$$

5.2 The Torsion Formula

✓ If the material is linear-elastic, then Hooke's law applies, $\tau = G\,\gamma$, and consequently a ***linear variation in shear strain*** leads to a corresponding ***linear variation in shear stress*** along any radial line on the cross section.

$$\gamma = \left(\frac{\rho}{c}\right)\gamma_{\max} \qquad \tau = G\ \gamma$$

$$\tau = \left(\frac{\rho}{c}\right)\tau_{\max}$$



Shear-stress distribution is a function of the radial position ρ of the element.

✓ The shear stress is always perpendicular to ρ .

$$T = \int_A \rho(\tau\, dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{\max}\ dA$$

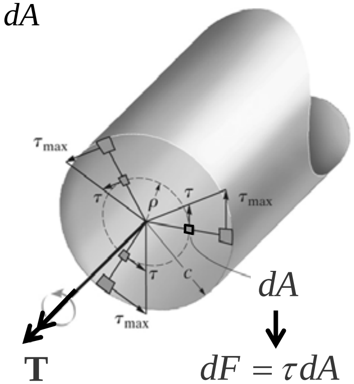
$$T = \frac{\tau_{\max}}{c}\int_A \rho^2 dA$$

➢ The integral depends only on the geometry of the shaft.

Polar moment of inertia

$$J = \int_A \rho^2 dA$$

$$\Rightarrow \quad \boxed{\tau_{\max} = \frac{Tc}{J}}$$



dA

$\downarrow$

$dF = \tau dA$

$\downarrow$

$dT = \rho dF = \rho\tau dA$

$$\boxed{\tau_{\max} = \frac{Tc}{J}}$$ Torsion formula

$\tau_{\max}$: the maximum shear stress in the shaft, which occurs at the outer surface

T : the resultant internal torque acting at the cross section

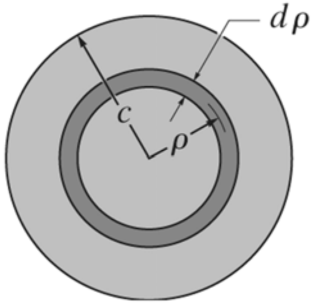
J : the polar moment of inertia of the cross-sectional area

c : the outer radius of the shaft

$$\tau = \left(\frac{\rho}{c}\right)\tau_{\max} \Rightarrow \boxed{\tau = \frac{T\rho}{J}}$$ Torsion formula

➢ It is used only if the shaft is circular and the material is homogeneous and behaves in a linear-elastic manner.

○ **Solid Shaft**



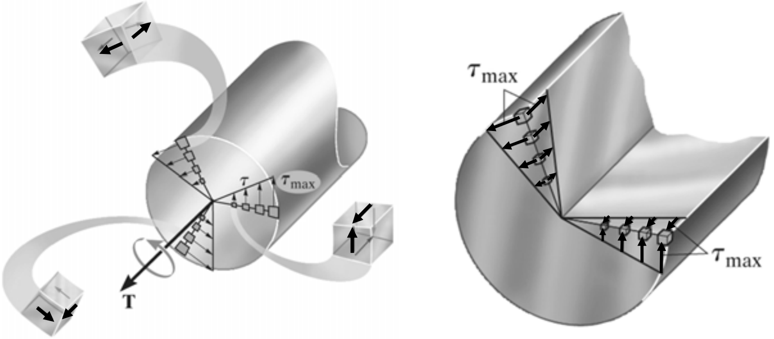
$$dA = 2\pi\rho\, d\rho$$

Polar moment of inertia

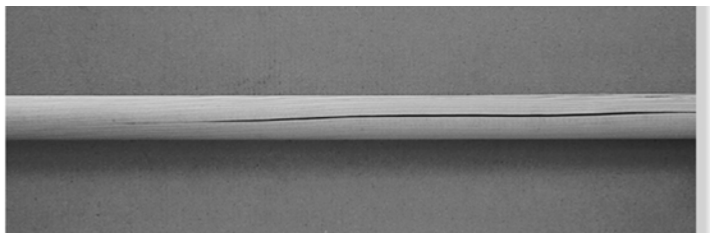
$$J = \int_A \rho^2 dA$$
$$= \int_0^c \rho^2 (2\pi\rho\, d\rho)$$
$$= 2\pi \int_0^c \rho^3 d\rho$$
$$= 2\pi \left(\frac{1}{4}\right)\rho^4 \Big|_0^c$$

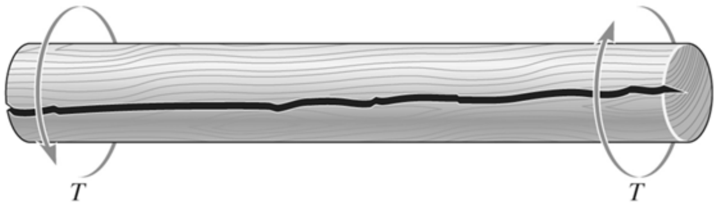
$$\boxed{J = \frac{\pi}{2}c^4}$$ Units: mm^4 or in^4

✓ J is a geometric property of the circular area and is always positive.



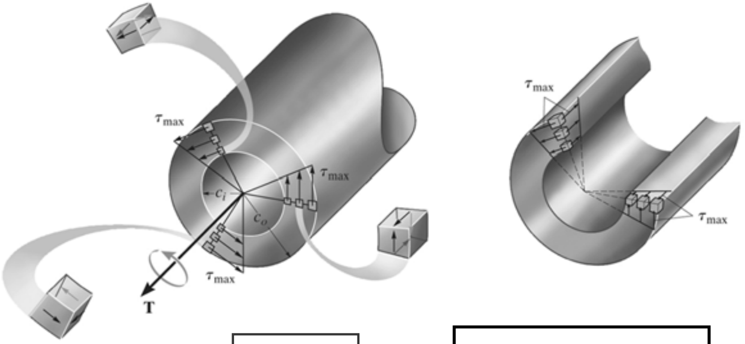
➢ The shear stress varies linearly along each radial line of the cross section of the shaft.
➢ Due to the complementary property of shear, equal shear stresses must also act on four of its adjacent faces.
➢ Not only does the internal torque T develop a linear distribution of shear stress along each radial line in the plane of the cross-sectional area, but also an associated shear-stress distribution is developed along an axial plane.





Failure of a wooden shaft due to torsion.

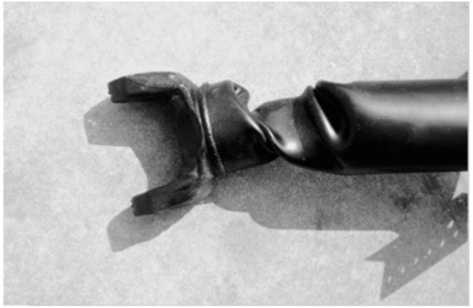
○ **Tubular Shaft**



$$\boxed{\tau = \frac{T\rho}{J}}$$

$$\boxed{J = \frac{\pi}{2}(c_o^4 - c_i^4)}$$

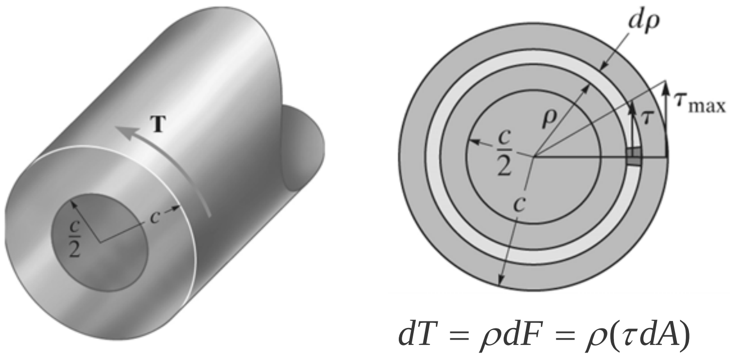
➢ Like the solid shaft, the shear stress distributed over the tube's cross-sectional area varies linearly along any radial line.
➢ The shear stress varies along an axial plane in this same manner.



This tubular drive shaft for a truck was subjected to an overload resulting in failure caused by yielding of the material .

EXAMPLE 5.1

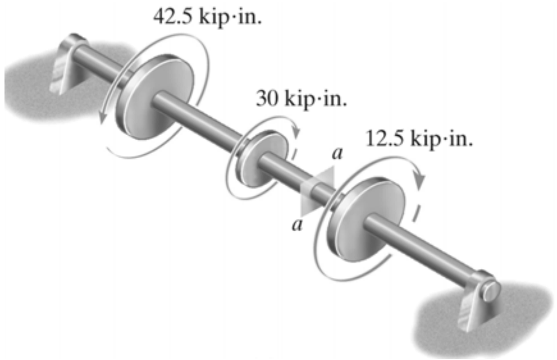
The solid shaft of radius c is subjected to a torque **T**. Determine the fraction of T that is resisted by the material contained within the outer region of the shaft, which has an inner radius of $c/2$ and outer radius c.

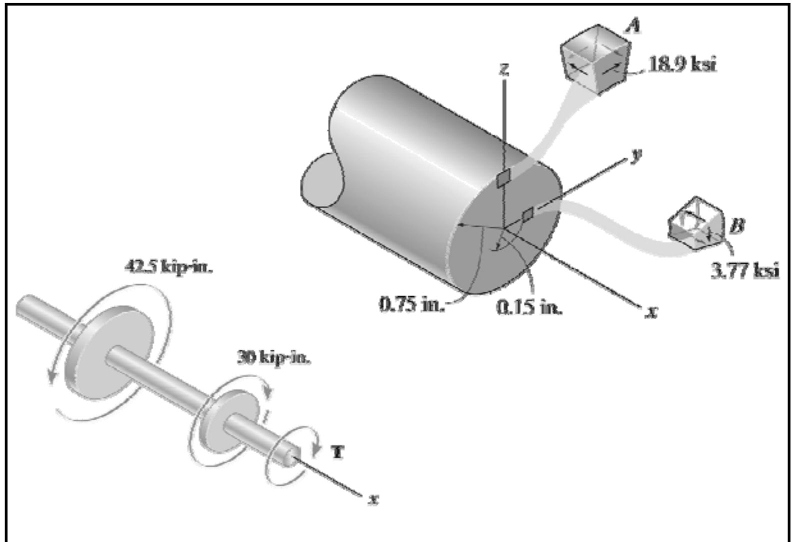


$$dT = \rho dF = \rho(\tau dA)$$

EXAMPLE 5.2

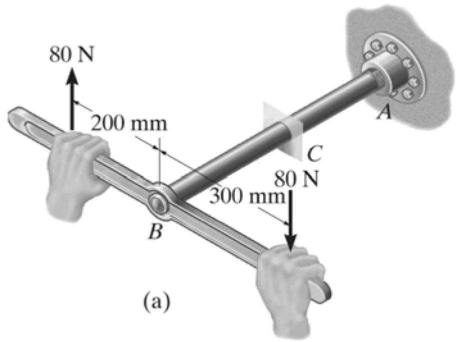
The shaft shown in Figure is supported by two bearings and is subjected to three torques. Determine the shear stress developed at points A and B, located at section a-a of the shaft.



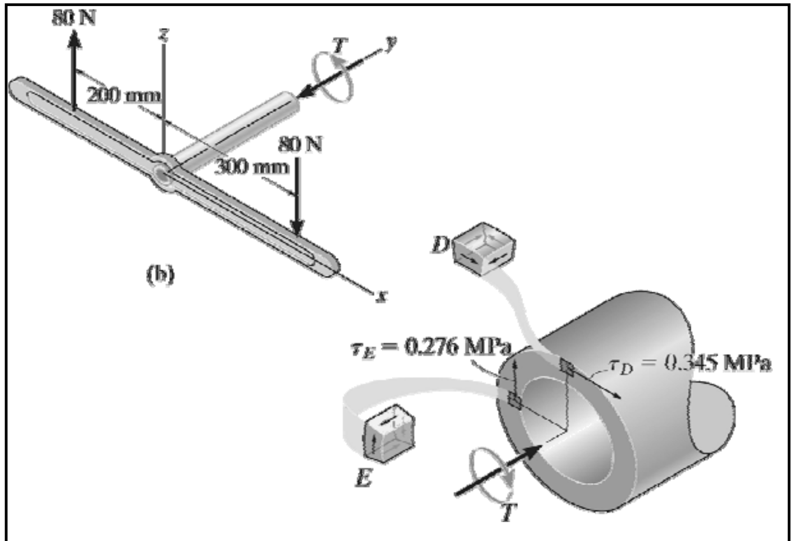


EXAMPLE 5.3

The pipe shown in Figure has an inner diameter of 80 mm and an outer diameter of 100 mm. If its end is tightened against the support at *A* using a torque wrench at *B*, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the 80-N forces are applied to the wrench.

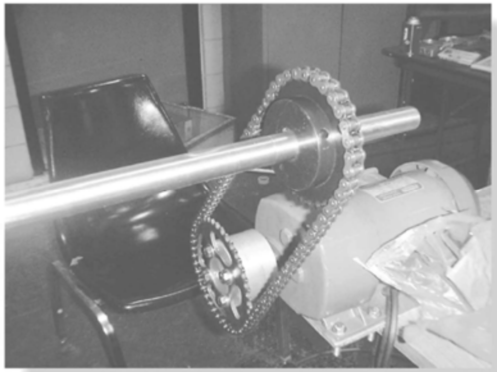


(a)



(b)

*5.3 Power Transmission



The chain drive transmits the torque developed by the electric motor to the shaft. The stress developed in the shaft depends upon the power transmitted by the motor and the rate of rotation of the connecting shaft.

- ***Power*** is defined as the work performed per unit of time.
- The work transmitted by a rotating shaft equals the **torque** applied times **the angle of rotation**.
- The instantaneous power

$$P = \frac{T\,d\theta}{dt} \qquad \text{Angular velocity} \quad \omega = \frac{d\theta}{dt}$$

$$\Rightarrow \boxed{P = T\omega}$$

- Units of stress
  - SI system
    $1\ \text{N}\cdot\text{m/s} = 1\ \text{W (watt)}$
  - U.S. Customary
    $1\ \text{hp (horsepower)} = 550\ \text{ft}\cdot\text{lb/s}$

- ***Frequency*** is defined as the number of revolutions or cycles the shaft makes per second

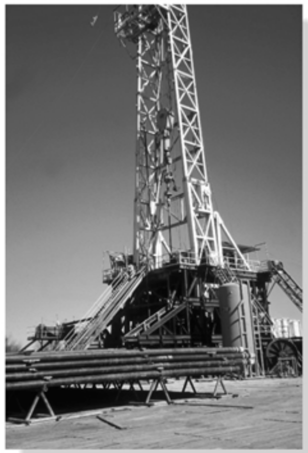
$$1\ \text{Hz (hertz)} = 1\ \text{cycle/s}$$

$$1\ \text{cycle} = 2\pi\ \text{rad}$$

$$\omega = 2\pi f$$

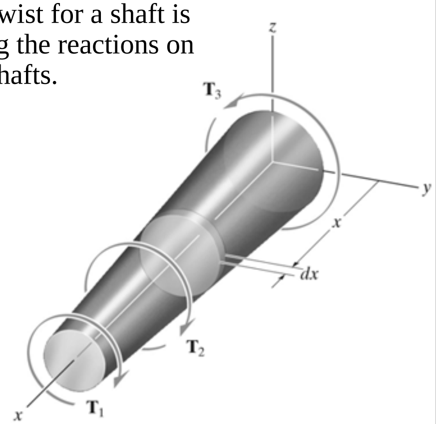
$$\Rightarrow \boxed{P = 2\pi f\, T}$$

5.4 Angle of Twist

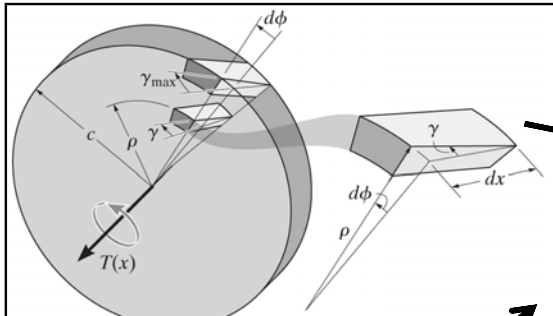


Oil wells are commonly drilled to depths exceeding a thousand meters. As a result, the total angle of twist of a string of drill pipe can be substantial and must be computed.

- The design of a shaft depends on restricting the amount of rotation or twist that may occur when the shaft is subjected to a torque.
- Computing the angle of twist for a shaft is important when analyzing the reactions on statically indeterminate shafts.



- homogeneous
- linear-elastic
- neglect the localized deformations



$$d\phi = \gamma \frac{dx}{\rho}$$

$$d\phi = \frac{T(x)}{J(x)G} dx$$

Hooke's law

$\gamma = \tau / G$

Torsion formula

$\tau = T(x)\rho / J(x)$

$$\gamma = \frac{T(x)\rho}{J(x)G}$$

$$\boxed{\phi = \int_0^L \frac{T(x)dx}{J(x)G}}$$

$$\boxed{\phi = \int_0^L \frac{T(x)dx}{J(x)G}}$$ Angle of Twist

ϕ : the angle of twist of one end of the shaft with respect to the other end, measured in radians

$T(x)$: the internal torque at the arbitrary position x

$J(x)$: the shaft's polar moment of inertia expressed as a function of position x

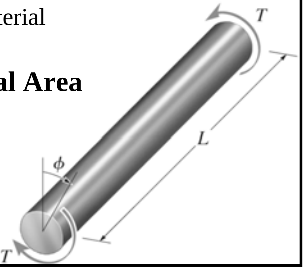
G : the shear modulus of elasticity for the material

○ **Constant Torque and Cross-Sectional Area**

$T(x) = T$

$J(x) = J$

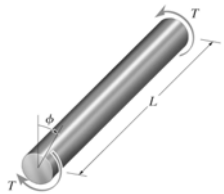
$$\boxed{\phi = \frac{TL}{JG}}$$

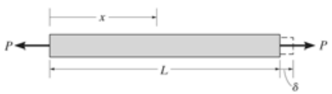


Torsion | **Axially loaded**

$$\phi = \int_0^L \frac{T(x)dx}{J(x)G} \longleftrightarrow \delta = \int_0^L \frac{P(x)\,dx}{A(x)E}$$

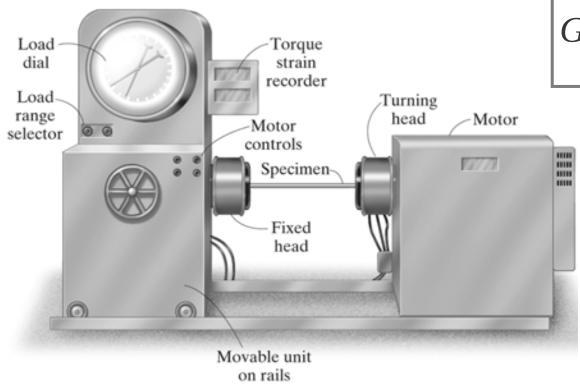
$$\phi = \frac{TL}{JG} \longleftrightarrow \delta = \frac{PL}{AE}$$





Determine the shear modulus of elasticity *G* of the material

$$\boxed{G = \frac{TL}{J\phi}}$$



When computing both the stress and the angle of twist of this soil auger, it is necessary to consider the variable loading which acts along its length.

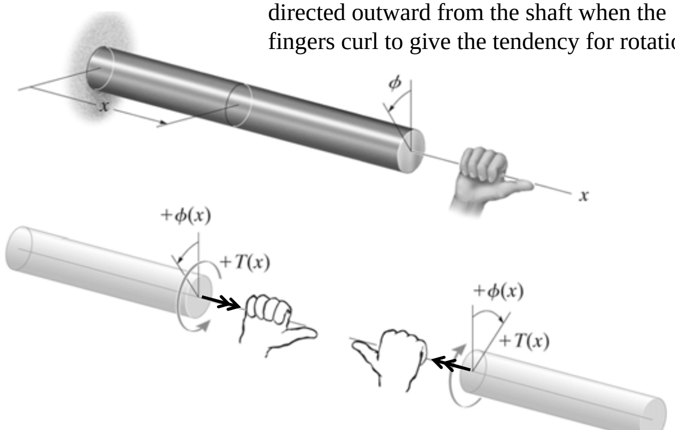
If the bar is subjected to several different torques, or the cross-sectional area or shear modulus changes abruptly from one region of the shaft to the next,

$$\boxed{\phi = \sum \frac{TL}{JG}}$$

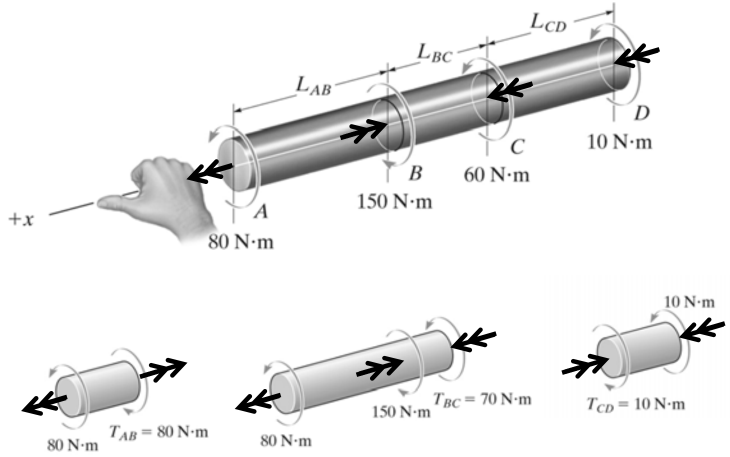
The angle of twist of one end of the shaft with respect to the other is then found from the vector addition of the angles of twist of each segment.

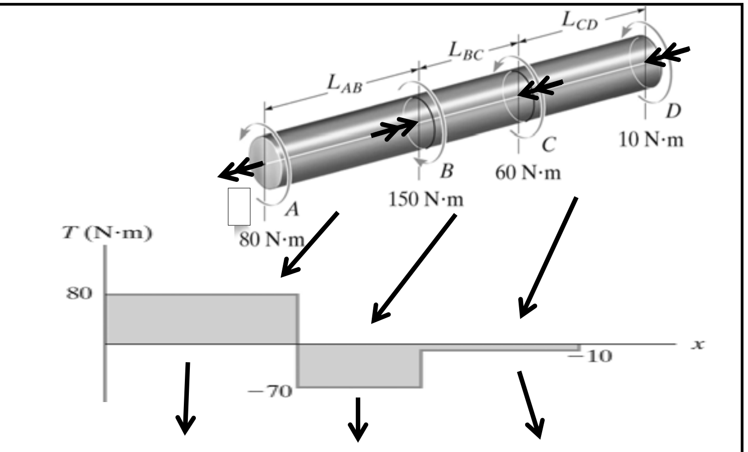
○ **Sign Convention**

➢ **Right-hand rule**: both the torque and angle will be positive, provided the thumb is directed outward from the shaft when the fingers curl to give the tendency for rotation.



Positive sign convention

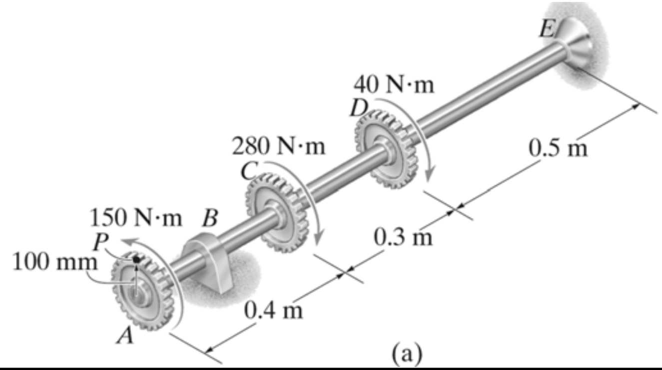


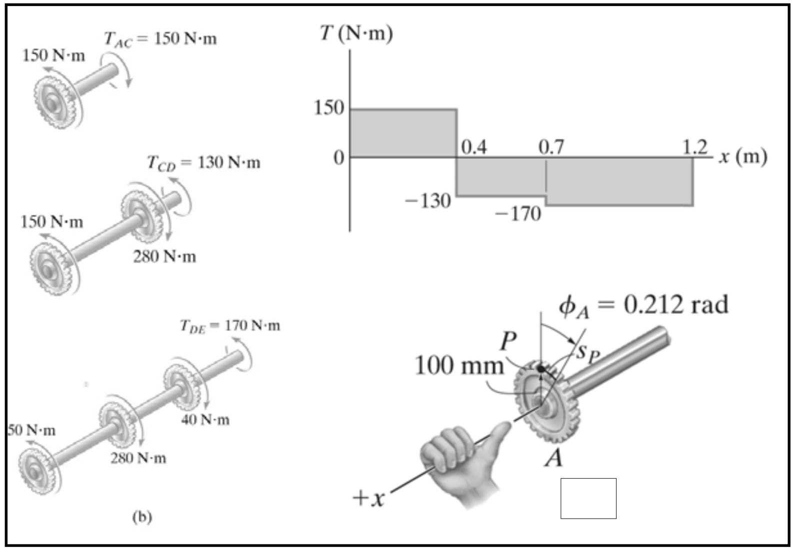


$$\phi_{A/D} = \frac{(+80N \cdot m)L_{AB}}{JG} + \frac{(-70N \cdot m)L_{BC}}{JG} + \frac{(-10N \cdot m)L_{CD}}{JG}$$

EXAMPLE 5.5

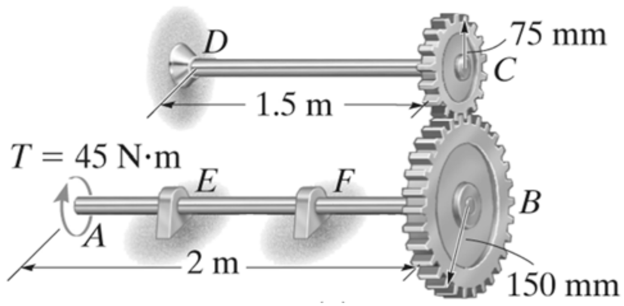
The gears attached to the fixed-end steel shaft are subjected to the torques shown in Figure. If the shear modulus of elasticity is 80 GPa and the shaft has a diameter of 14 mm, determine the displacement of the tooth *P* on gear *A*. The shaft turns freely within the bearing at *B*.

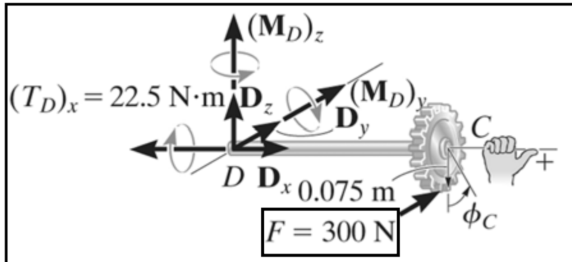




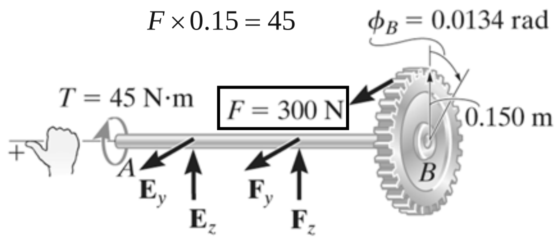
EXAMPLE 5.6

The two solid steel shafts shown in Figure are coupled together using the meshed gears. Determine the angle of twist of end *A* of shaft *AB* when the torque *T* = 45 N ・m is applied. Take G = 80 Gpa. Shaft *AB* is free to rotate within bearings *E* and *F*, whereas shaft *DC* is fixed at *D*. Each shaft has a diameter of 20 mm.



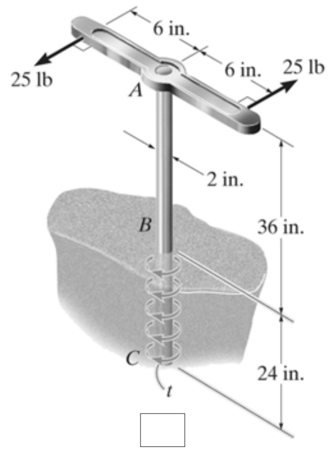


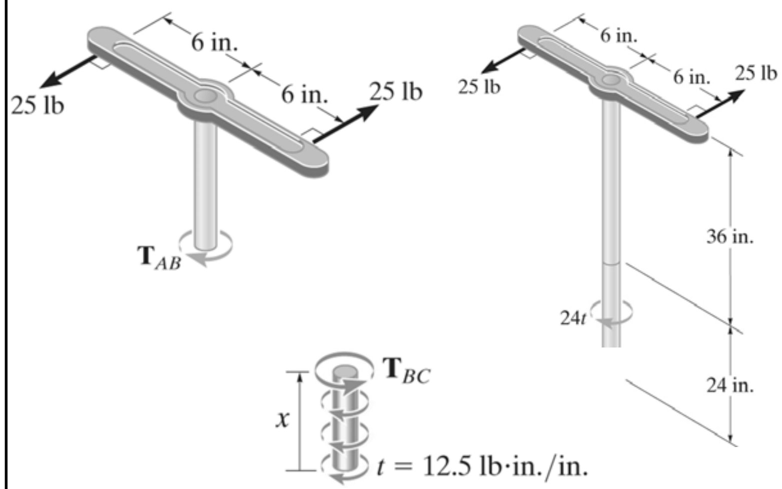
$F \times 0.15 = 45$



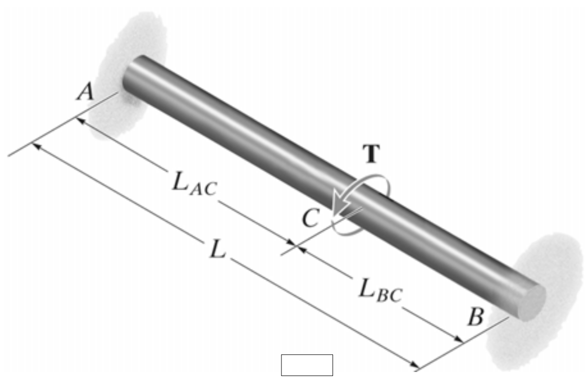
EXAMPLE 5.7

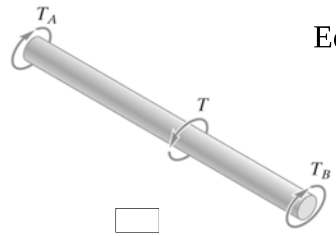
The 2-in.-diameter solid cast-iron post shown in Figure is buried 24 in. in soil. If a torque is applied to its top using a rigid wrench, determine the maximum shear stress in the post and the angle of twist at its top. Assume that the torque is about to turn the post, and the soil exerts a uniform torsional resistance of t lb • in./in. along its 24-in. buried length. $G = 5.5(10^3)$ ksi.





5.5 Statically Indeterminate Torque-Loaded Members

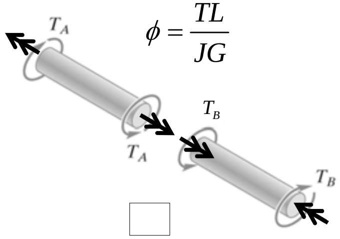




Equilibrium equation

$$\sum M_x = 0$$

$$T - T_A - T_B = 0 \qquad \textbf{(1)}$$



$$\phi = \frac{TL}{JG}$$

Compatibility equation

$$\phi_{A/B} = 0$$

$$\frac{T_A L_{AC}}{JG} - \frac{T_B L_{BC}}{JG} = 0 \qquad \textbf{(2)}$$

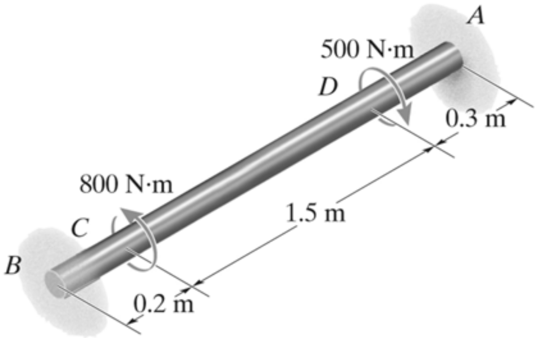
Solve (1) and (2) $\qquad L = L_{AC} + L_{BC}$

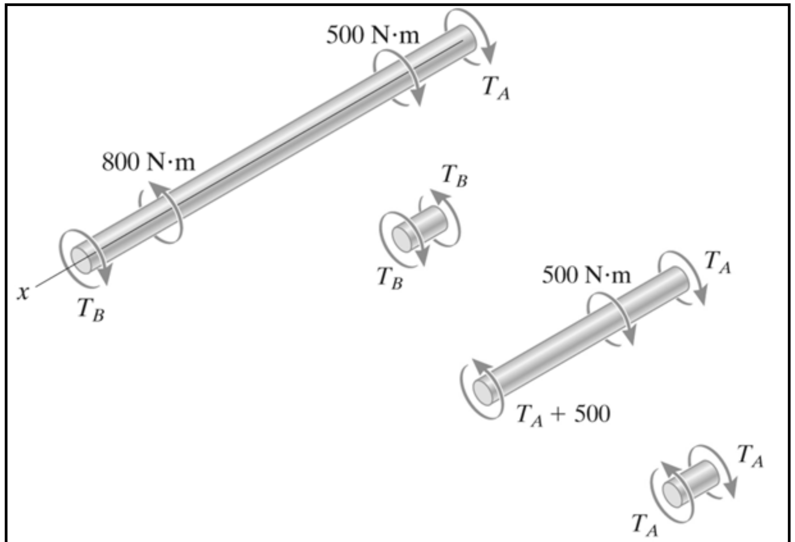
$$T_A = T\left(\frac{L_{BC}}{L}\right)$$

$$T_B = T\left(\frac{L_{AC}}{L}\right)$$

EXAMPLE 5.8

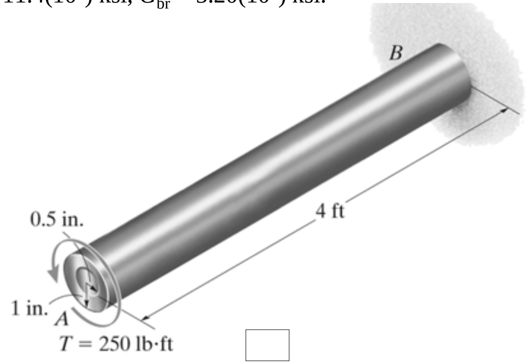
The solid steel shaft shown in Figure has a diameter of 20 mm. If it is subjected to the two torques, determine the reactions at the fixed supports *A* and *B*.

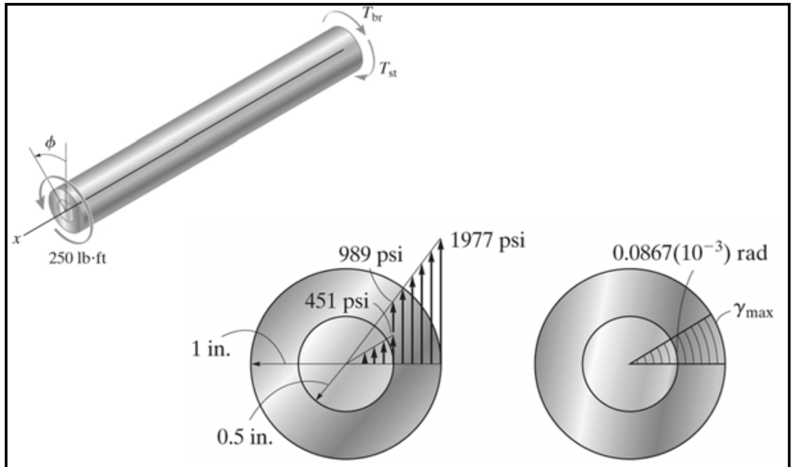




EXAMPLE 5.9

The shaft shown in Figure is made from a steel tube, which is bonded to a brass core. If a torque of $T = 250$ lb • ft is applied at its end, plot the shear-stress distribution along a radial line of its cross-sectional area. Take $G_{st} = 11.4(10^3)$ ksi, $G_{br} = 5.20(10^3)$ ksi.

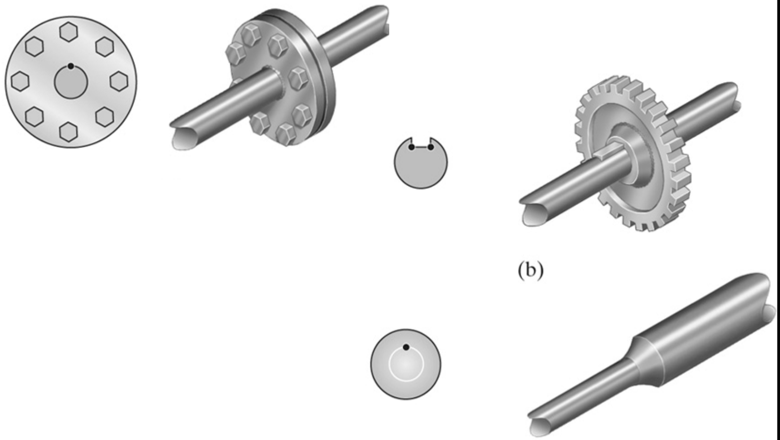


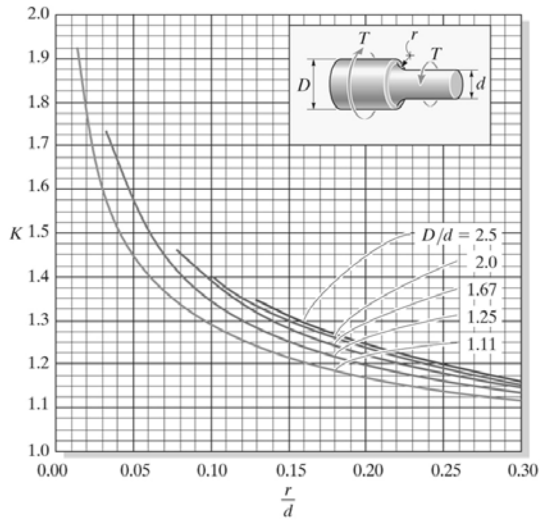


Shear–stress distribution

Shear–strain distribution

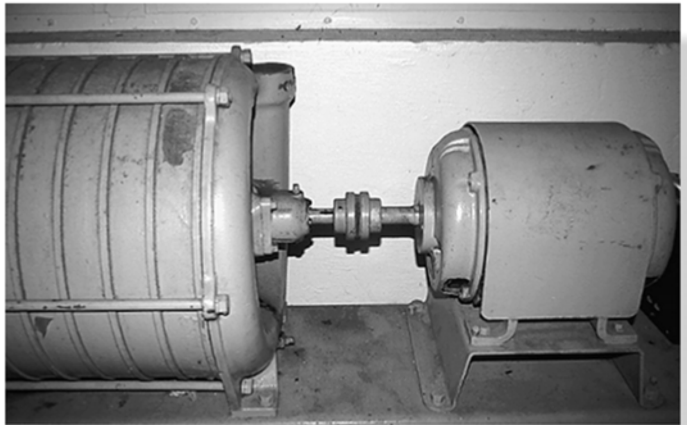
*5.8 Stress Concentration





$$\tau_{\max} = K\frac{Tc}{J}$$

K : torsional stress-concentration factor



Stress concentrations can arise at the coupling of these shafts, and this must be taken into account when the coupling is designed.

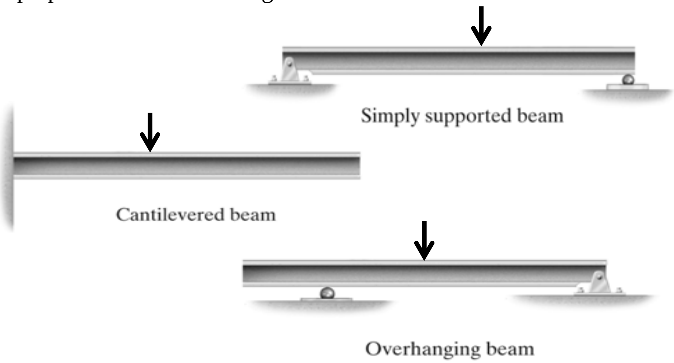
Chapter 6
Bending

Beams are important structural members used in building construction. Their design is often based upon their ability to resist bending stress, which forms the subject matter of this chapter.

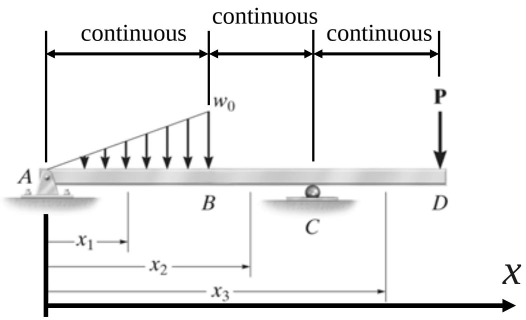
6.1 Shear and Moment Diagrams

○ **Beams**

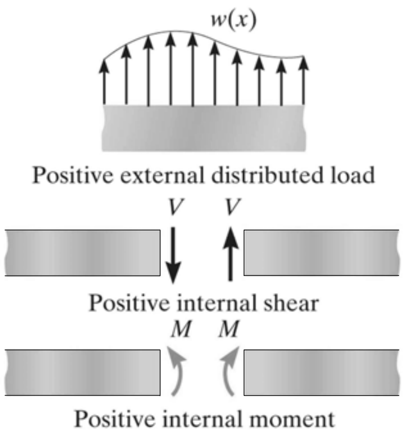
➢ Members that are slender and support loadings that are applied perpendicular to their longitudinal axis



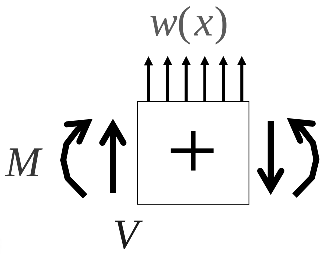
➢ Beams develop an internal shear force and bending moment that vary from point to point along the axis of the beam.
➢ In order to properly design a beam it is first necessary to determine the maximum shear and moment in the beam.
➢ Express ***V*** and ***M*** as functions of the arbitrary position ***x*** along the beam's axis, which can be plotted and represented by graphs called ***shear and moment diagrams***.



○ Sign Convention

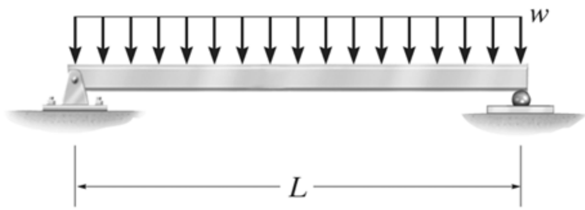


Beam sign convention



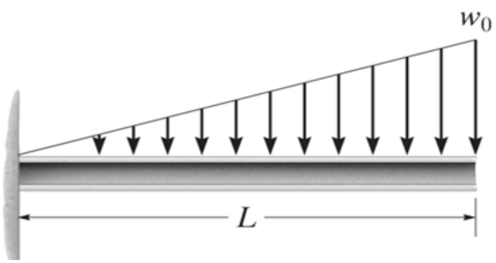
EXAMPLE 6.1

Draw the shear and moment diagrams for the beam shown in Figure.



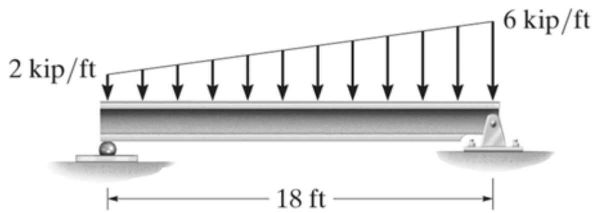
EXAMPLE 6.2

Draw the shear and moment diagrams for the beam shown in Figure.



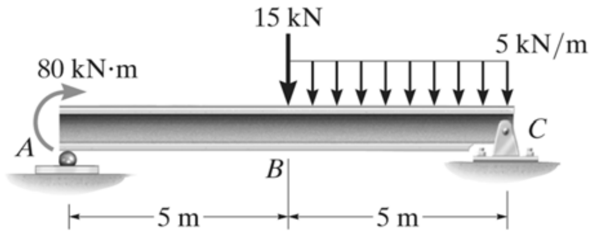
EXAMPLE 6.3

Draw the shear and moment diagrams for the beam shown in Figure.



EXAMPLE 6.4

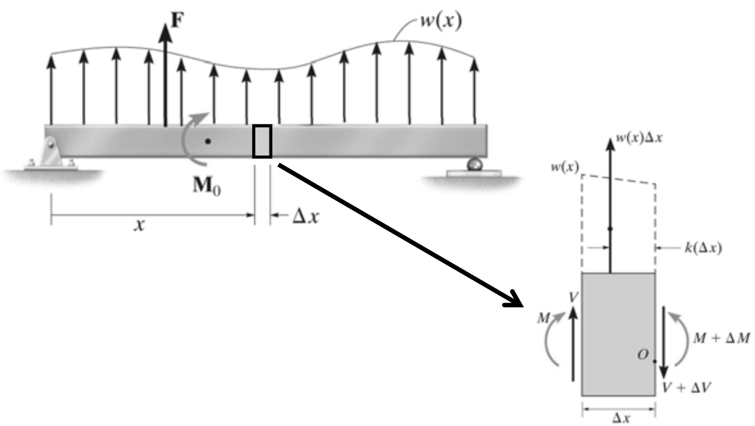
Draw the shear and moment diagrams for the beam shown in Figure.



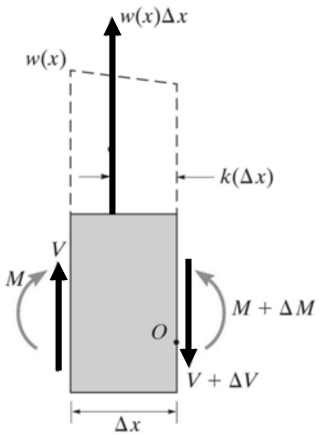
6.2 Graphical Method for Constructing Shear and Moment Diagrams

Failure of this table occurred at the brace support on its right side. If drawn, the bending moment diagram for the table loading would indicate this to be the point of maximum internal moment.

○ Regions of Distributed Load



Free-body diagram of segment Δx



Free-body diagram of segment Δx

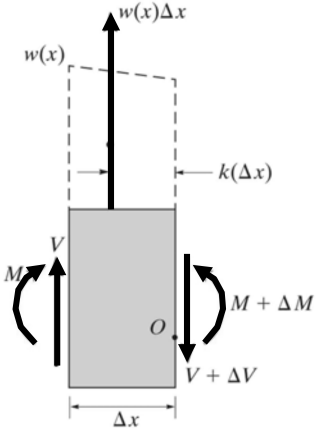
$$+\uparrow \quad \sum F_y = 0$$

$$V + w(x)\Delta x - (V + \Delta V) = 0$$

$$\Rightarrow \quad \Delta V = w(x)\Delta x$$

As $\Delta x \to 0$

$$\boxed{\frac{dV}{dx} = w(x)}$$



Free-body diagram of segment Δx

$$\circlearrowleft + \quad \sum M_o = 0$$

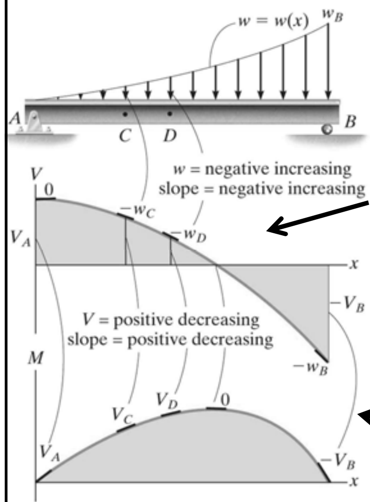
$$-V\Delta x - M - w(x)\Delta x\left[k(\Delta x)\right]$$

$$+(M + \Delta M) = 0$$

$$\Rightarrow \quad \Delta M = V\Delta x + w(x)k(\Delta x)^2$$

As $\Delta x \to 0$

$$\boxed{\frac{dM}{dx} = V}$$

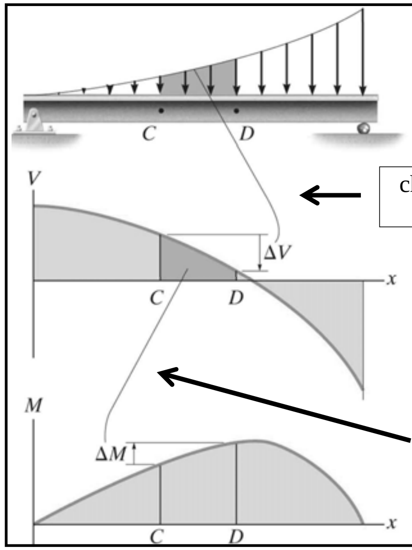


$$\boxed{\frac{dV}{dx} = w(x)}$$

slope of shear diagram at each point = distributed load intensity at each point

$$\boxed{\frac{dM}{dx} = V}$$

slope of moment diagram at each point = shear at each point



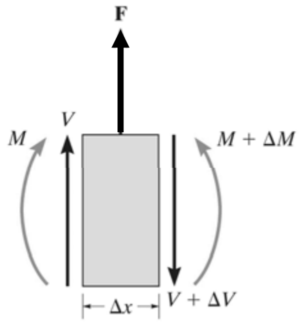
$$\boxed{\Delta V = \int w(x)\ dx}$$

change in shear = area under distributed loading

$$\boxed{\Delta M = \int V(x)\ dx}$$

change in moment = area under shear diagram

○ Regions of Concentrated Force and Moment



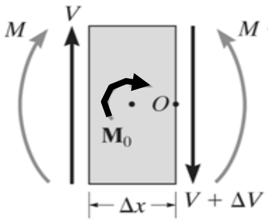
$$+\uparrow \quad \sum F_y = 0$$

$$V + F - (V + \Delta V) = 0$$

$$\Rightarrow \quad \Delta V = F$$

When **F** acts *upward* on the beam, ΔV is *positive* so the shear will “jump” *upward*. Likewise, if **F** acts *downward*, the jump (ΔV) will be *downward*.

○ Regions of Concentrated Force and Moment



$$\circlearrowright + \quad \sum M_o = 0$$

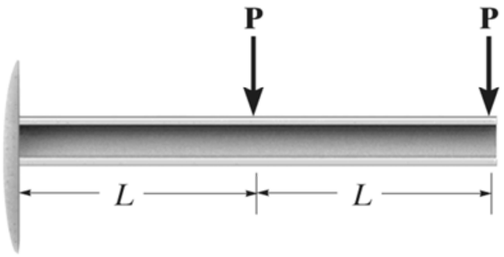
$$(M + \Delta M) - M_0 - V\Delta x - M = 0$$

$$\Rightarrow \quad \Delta M = M_0$$

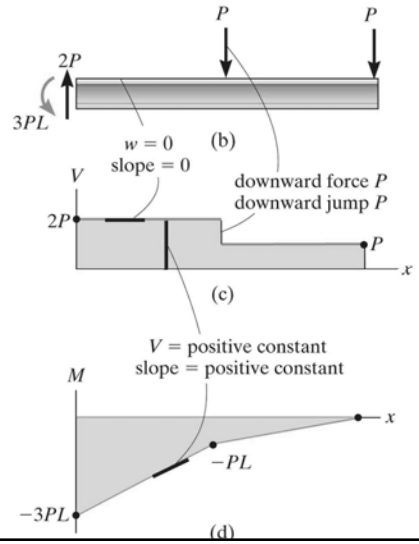
If $\mathbf{M}_0$ is applied *clockwise*, ΔM is positive so the moment diagram will "jump" *upward*. Likewise, when $\mathbf{M}_0$ acts *counterclockwise*, the jump (ΔM) will be *downward*.

EXAMPLE 6.5

Draw the shear and moment diagrams for the beam in Figure.

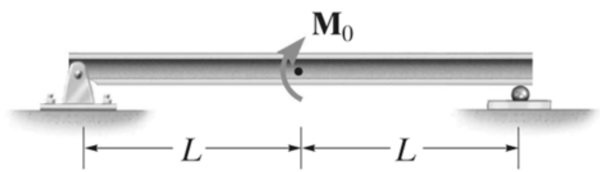


EXAMPLE 6.5

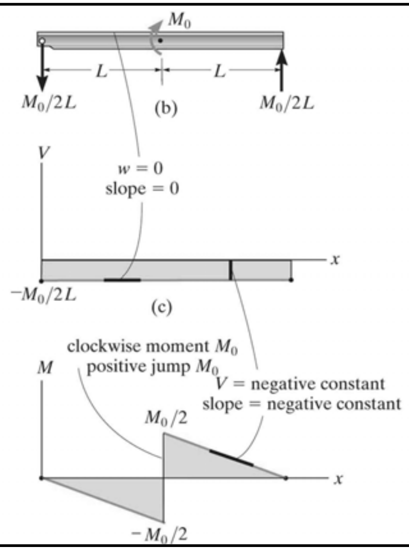


EXAMPLE 6.6

Draw the shear and moment diagrams for the beam in Figure.

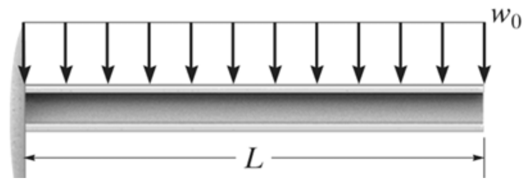


EXAMPLE 6.6

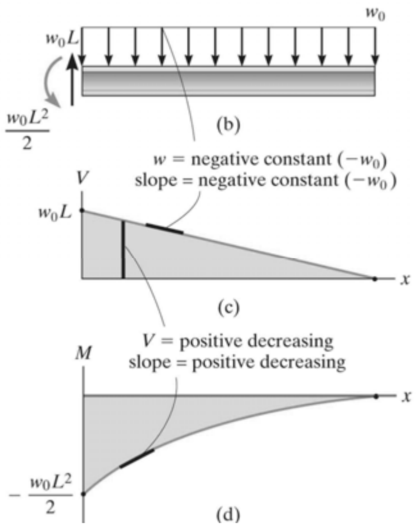


EXAMPLE 6.7(a)

Draw the shear and moment diagrams for each of the beams in Figures.

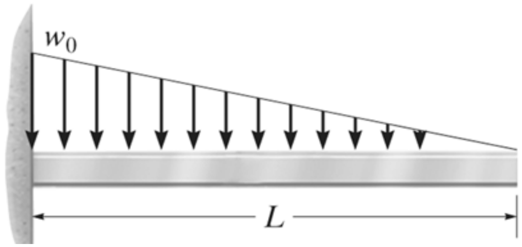


EXAMPLE 6.7(a)

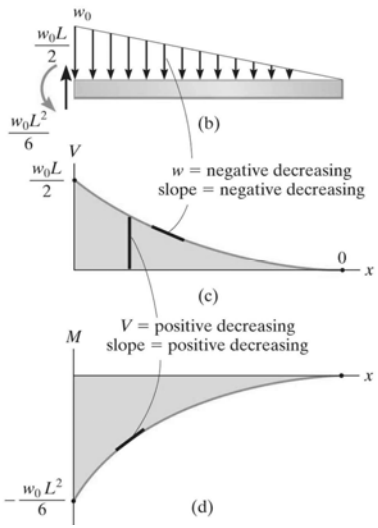


EXAMPLE 6.7(b)

Draw the shear and moment diagrams for each of the beams in Figures.

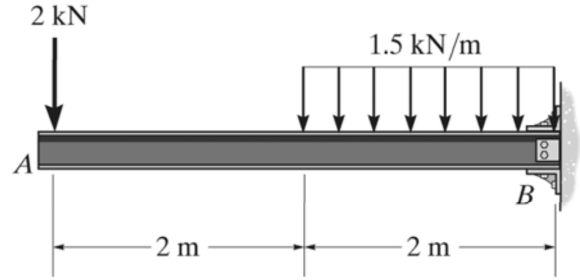


EXAMPLE 6.7(b)

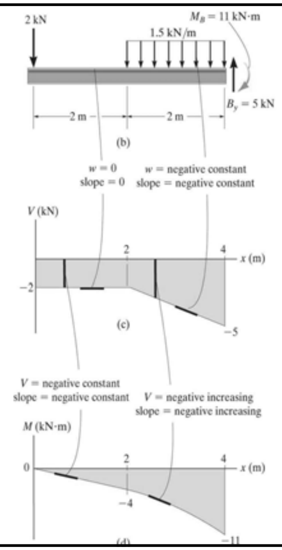


EXAMPLE 6.8

Draw the shear and moment diagrams for the cantilever beam in Figure.

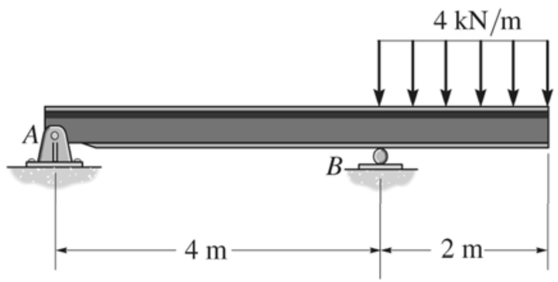


EXAMPLE 6.8

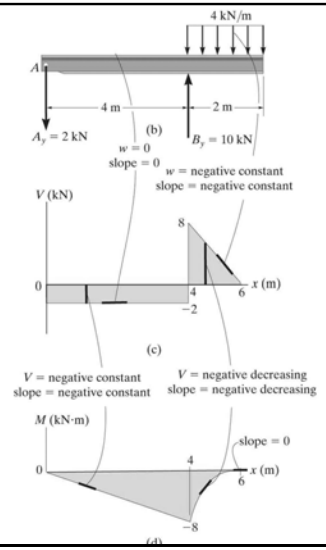


EXAMPLE 6.9

Draw the shear and moment diagrams for the overhang beam in Figure.

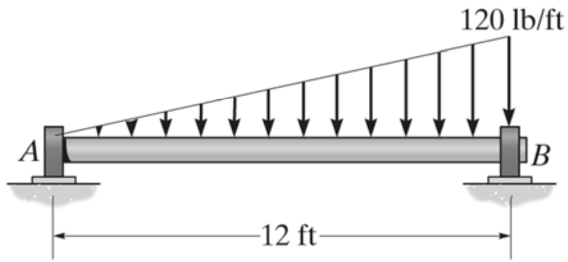


EXAMPLE 6.9

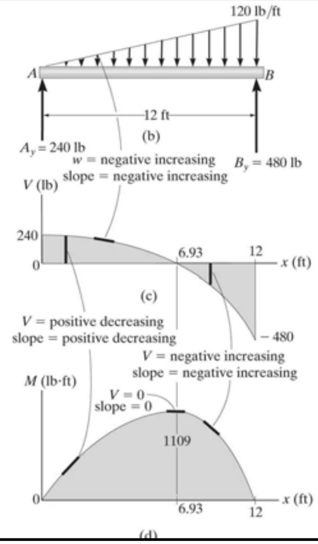


EXAMPLE 6.10

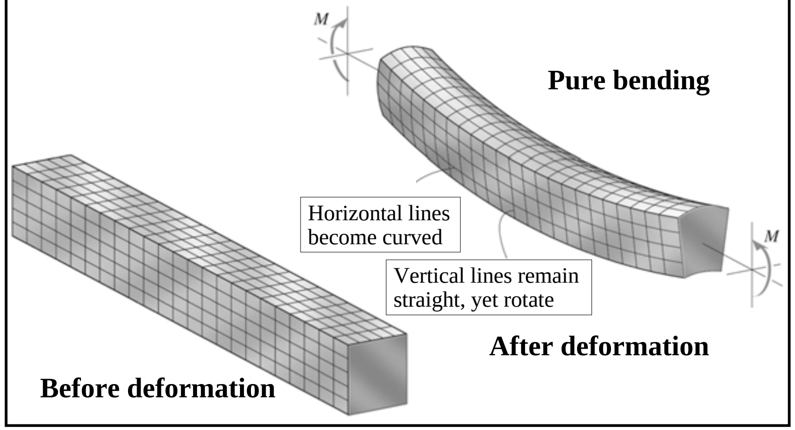
The shaft in Figure is supported by a thrust bearing at *A* and a journal bearing at *B*. Draw the shear and moment diagrams.

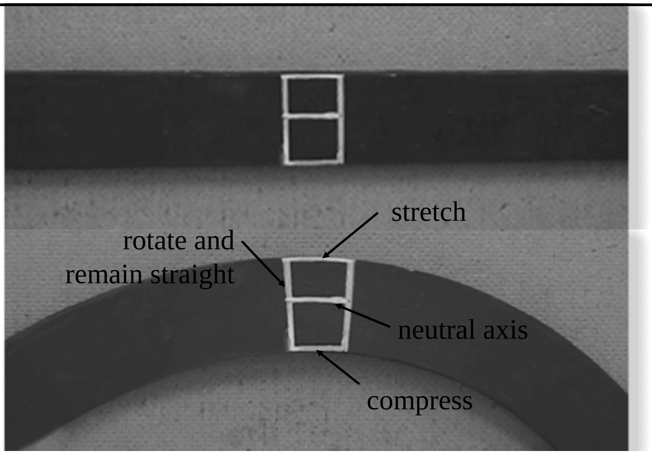


EXAMPLE 6.10

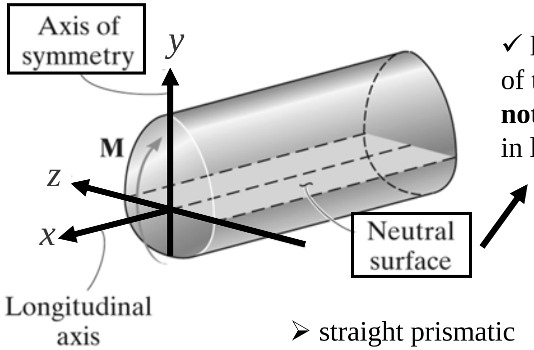


6.3 Bending Deformation of a Straight Member





Note the distortion of the lines due to bending of this rubber bar. The top line stretches, the bottom line compresses, and the center line remains the same length. Furthermore the vertical lines rotate and yet remain straight.



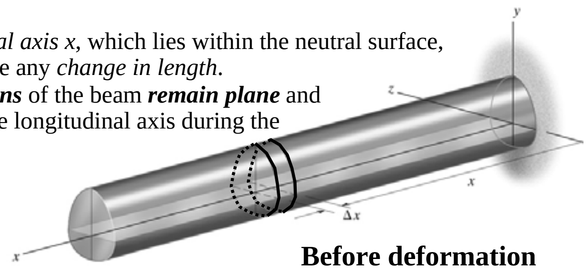
✓ longitudinal fibers of the material will **not** undergo a change in length.

- straight prismatic
- homogeneous
- having a cross-sectional area that is symmetrical with respect to an axis, and the bending moment is applied about an axis perpendicular to this axis of symmetry.

Assumptions

(1) The *longitudinal axis x*, which lies within the neutral surface, does not experience any *change in length*.

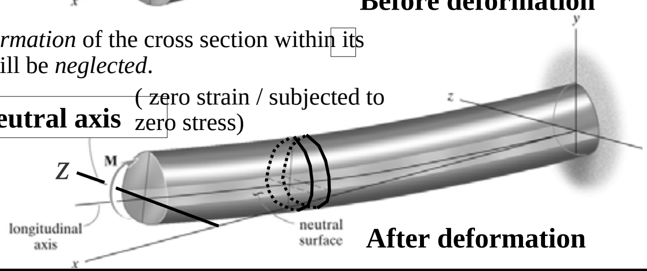
(2) All ***cross sections*** of the beam ***remain plane*** and perpendicular to the longitudinal axis during the deformation.



Before deformation

(3) Any *deformation* of the cross section within its own plane will be *neglected*.

Neutral axis (zero strain / subjected to zero stress)



After deformation

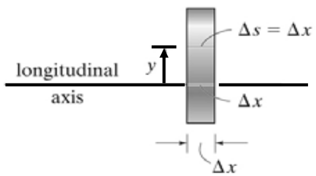
Normal strain along Δs

$$\varepsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

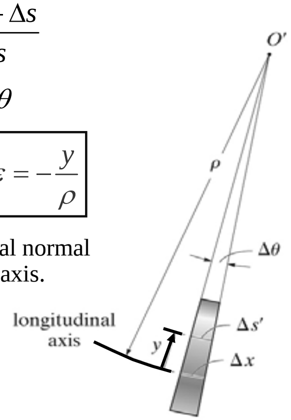
$$\Delta s = \Delta x = \rho\, \Delta\theta \qquad \Delta s' = (\rho - y)\Delta\theta$$

$$\varepsilon = \lim_{\Delta\theta \to 0} \frac{(\rho - y)\Delta\theta - \rho\, \Delta\theta}{\rho\, \Delta\theta} \Rightarrow \boxed{\varepsilon = -\frac{y}{\rho}}$$

✓ For any specific cross section, the longitudinal normal strain will vary linearly with y from the neutral axis.



Undeformed element



Deformed element

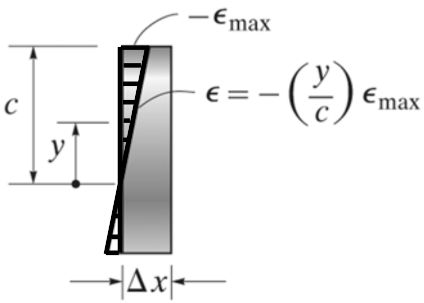
$$\boxed{\varepsilon = -\frac{y}{\rho}}$$

✓ A contraction $(-\varepsilon)$ will occur in fibers located above the neutral axis $(+y)$, whereas elongation$(+\varepsilon)$ will occur in fibers located below the axis $(-y)$.

$$\varepsilon_{\max} = c/\rho$$

$$\frac{\varepsilon}{\varepsilon_{\max}} = -\frac{y/\rho}{c/\rho}$$

$$\Rightarrow \quad \varepsilon = -\left(\frac{y}{c}\right)\varepsilon_{\max}$$



Normal strain distribution

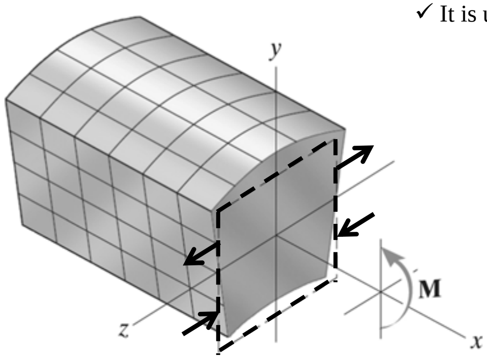
✓ Provided only a moment is applied to the beam, then this moment causes a *normal stress only* in the longitudinal or *x* direction.
✓ All the other components of normal and shear stress are zero, since the beam's surface is free of any other load.

✓ It is uniaxial state of stress.

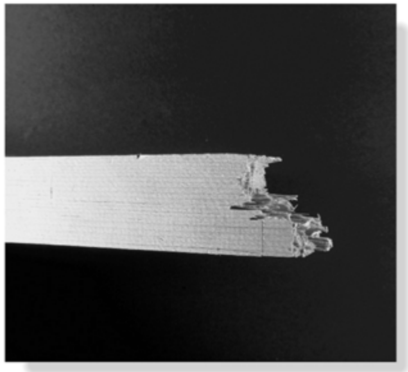
$$\varepsilon_x \quad (\sigma_x = E\varepsilon_x)$$

$$\varepsilon_y = -\nu\varepsilon_x$$

$$\varepsilon_z = -\nu\varepsilon_x$$



6.4 The Flexure Formula



This wood specimen failed in bending due to its fibers being crushed at its top and torn apart at its bottom.

○ Linear variation of normal strain (Plane remains plane.)

$$\varepsilon = -\left(\frac{y}{c}\right)\varepsilon_{\max}$$

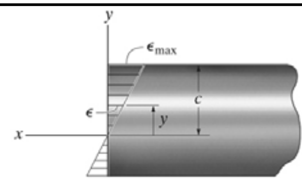
○ Linear-elastic manner

$$\sigma = E\varepsilon$$

○ A linear variation of normal strain must then be the consequence of a linear variation in normal stress.

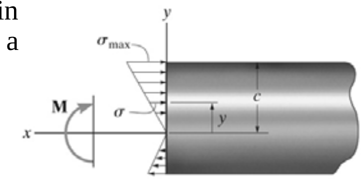
$$\sigma = -\left(\frac{y}{c}\right)\sigma_{\max}$$

✓ This equation represents the stress distribution over the cross-sectional area.



Normal strain variation
(profile view)

(a)



Bending stress variation
(profile view)

$$F_R = \Sigma\ F_x$$

$$0 = \int_A dF = \int_A \sigma dA$$

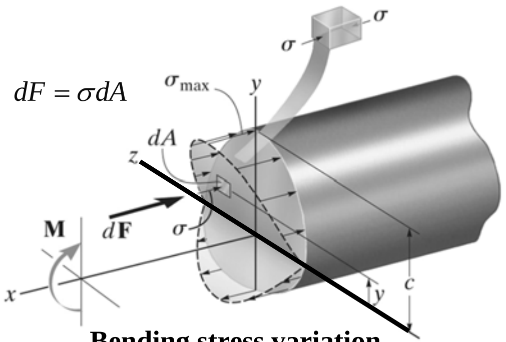
$$= \int_A -\left(\frac{y}{c}\right)\sigma_{\max} dA$$

$$= -\frac{\sigma_{\max}}{c}\int_A y dA$$

$$\because\ \frac{\sigma_{\max}}{c} \neq 0$$

$$\therefore\ \boxed{\int_A y dA = 0}$$

$$dF = \sigma dA$$



Bending stress variation

✓ This condition can only be satisfied if the ***neutral axis*** is also the horizontal ***centroidal axis*** for the cross section.
✓ Once the centroid for the member's cross-sectional area is determined, the location of neutral axis is known.

$$(M_R)_z = \Sigma\ M_z$$

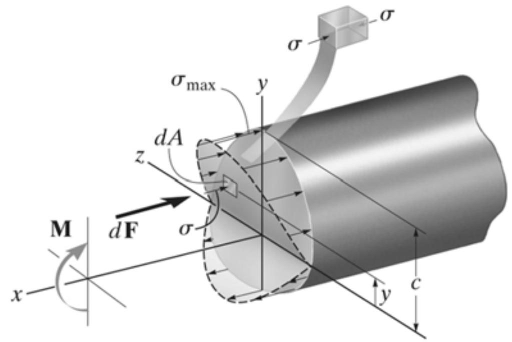
$$M = \int_A y dF = \int_A y(\sigma dA) = \int_A y\left(\frac{y}{c}\sigma_{\max}\right)dA$$

$$M = \frac{\sigma_{\max}}{c}\int_A y^2 dA$$

Moment of inertia

$$I = \int_A y^2 dA$$

$$\boxed{\sigma_{\max} = \frac{Mc}{I}}$$



Bending stress variation

$$\boxed{\sigma_{\max} = \frac{Mc}{I}}$$ Flexure formula

$\sigma_{\max}$: the maximum normal stress in the member, which occurs at a point on the cross-sectional area farthest away from the neutral axis

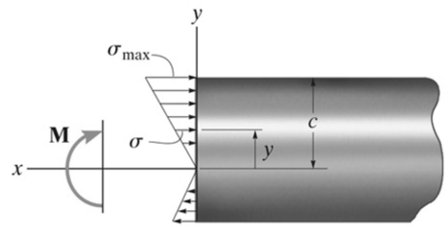
M : the resultant internal moment computed about the neutral axis of the cross section

I : the moment of inertia of the cross-sectional area computed about the neutral axis

c : the perpendicular distance from the neutral axis to a point farthest away from the neutral axis

Since $\dfrac{\sigma_{\max}}{c} = \dfrac{-\sigma}{y}$

$$\sigma_{\max} = \frac{-\sigma c}{y}$$



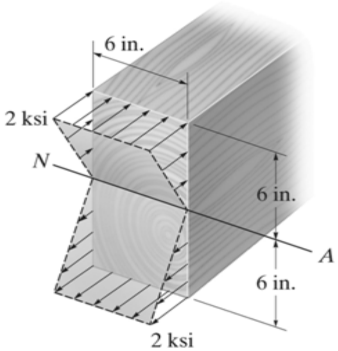
Bending stress variation
(profile view)

$$\Rightarrow\ \boxed{\sigma = -\frac{My}{I}}$$ Flexure formula

✓ The negative sign is necessary since it agrees with the established x, y, z axes. By the right-hand rule, M is positive along the $+z$ axis, y is positive upward, and σ therefore must be negative (compressive) since it acts in the negative x direction.

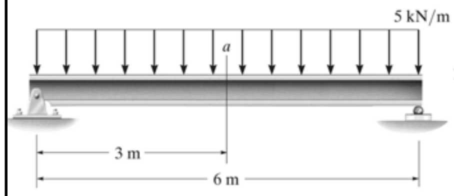
EXAMPLE 6.11

A beam has a rectangular cross section and is subjected to the stress distribution shown in Figure. Determine the internal moment **M** at the section caused by the stress distribution (a) using the flexure formula, (b) by finding the resultant of the stress distribution using basic principles.

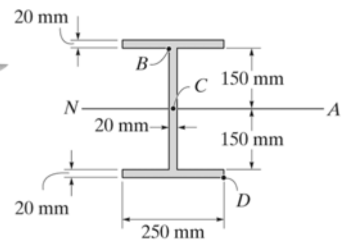


EXAMPLE 6.12

The simply supported beam in Figure (a) has the cross-sectional area shown in Figure (b). Determine the absolute maximum bending stress in the beam and draw the stress distribution over the cross section at this location.



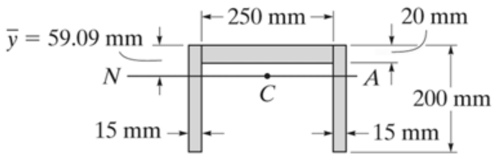
(a)



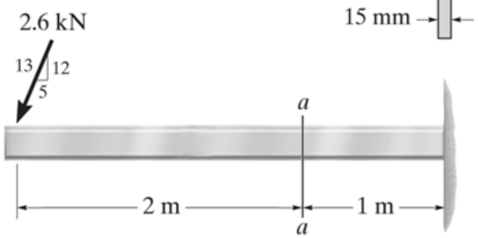
(b)

EXAMPLE 6.13

The beam shown in Figure (a) has a cross-sectional area in the shape of a channel, Figure (b). Determine the maximum bending stress that occurs in the beam at section a-a.



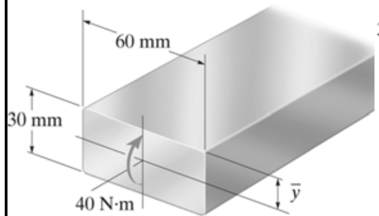
(b)



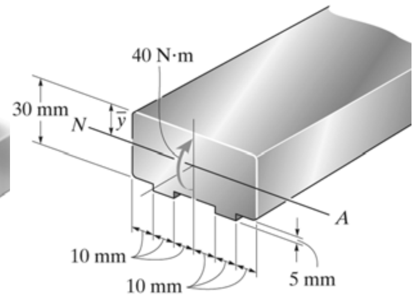
(a)

EXAMPLE 6.14

The member having a rectangular cross section, Figure (a), is designed to resist a moment of 40 N-m. In order to increase its strength and rigidity, it is proposed that two small ribs be added at its bottom, Figure (b). Determine the maximum normal stress in the member for both cases.

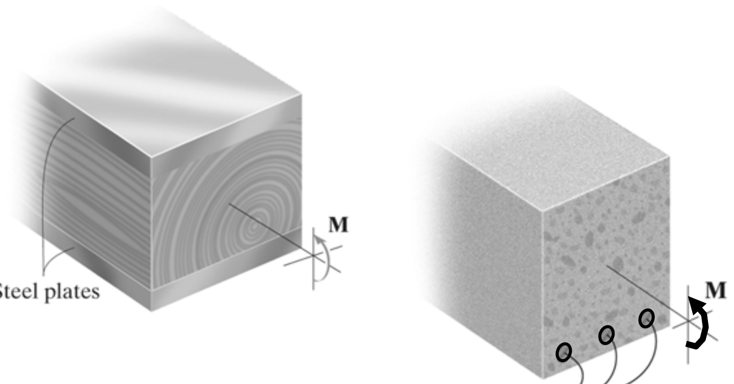


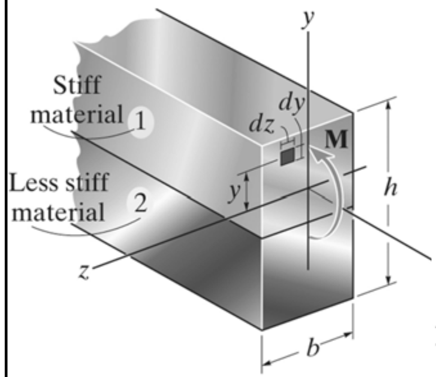
(a)

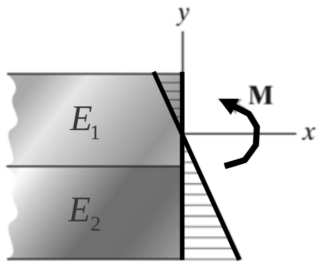


(b)

6.6 Composite Beams

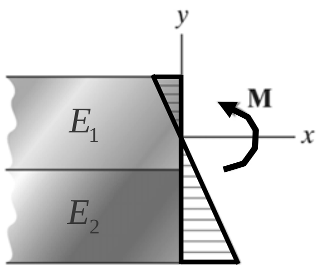




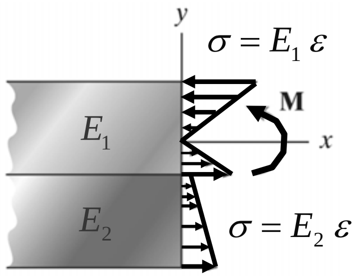


Normal strain variation

✓ The total cross-sectional area remains plane after bending.



Normal strain variation

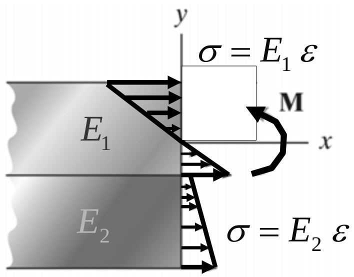


Bending stress variation

$$\sigma = E\varepsilon$$

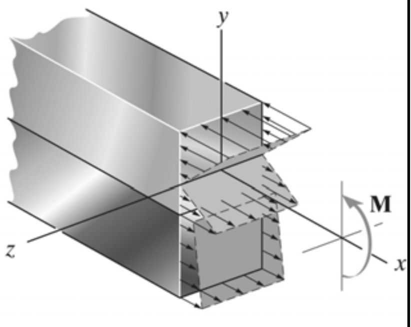
✓ If the material has linear-elastic behavior, Hooke's law applies.

✓ If material 1 is stiffer than material 2, e.g., steel versus rubber, most of the load will be carried by material 1, since $E_1 > E_2$.



Bending stress variation

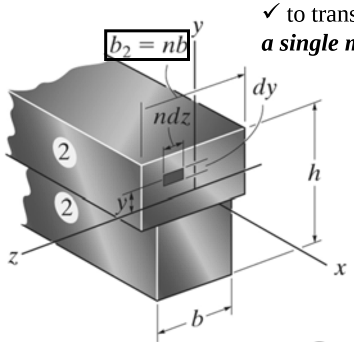
✓ Location of the neutral axis, and determination of the maximum stress in the beam, using this stress distribution, can be based on a trial-and-error procedure.



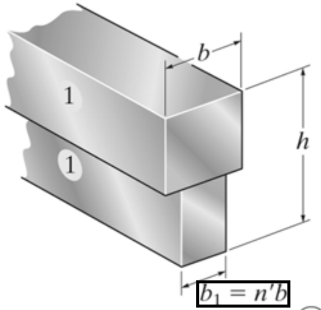
Bending stress variation

○ Transformed-section method

✓ to transform the beam into one made of ***a single material***



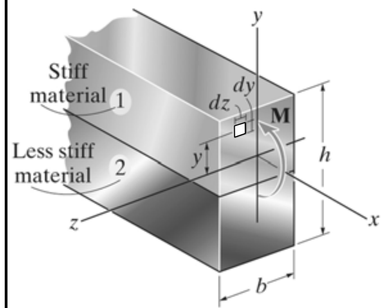
Beam transformed to material 2

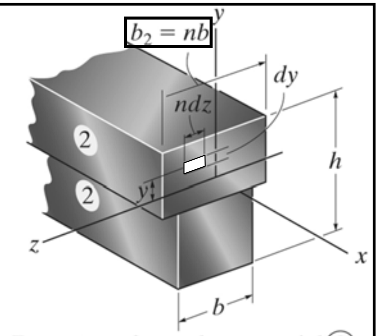


Beam transformed to material 1

✓ The height h remains the same, and the width of material 1 is changed to nb. $n = E_1 / E_2$

✓ The height h remains the same, and the width of material 2 is changed to $n'b$. $n' = E_2 / E_1$



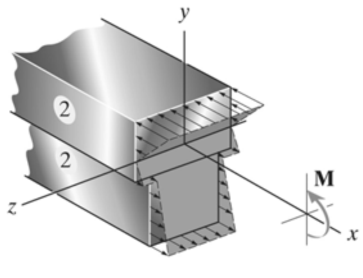


Beam transformed to material 2

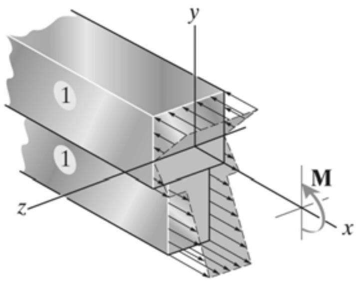
$$dF = \sigma dA = (E_1\varepsilon) dz dy \qquad dF' = \sigma' dA' = (E_2\varepsilon)(ndz) dy$$

Equivalent force $dF = dF'$

$$E_1 \varepsilon dz dy = E_2 \varepsilon n dz dy \quad \Rightarrow \quad \boxed{n = \frac{E_1}{E_2}}$$ Transformation factor



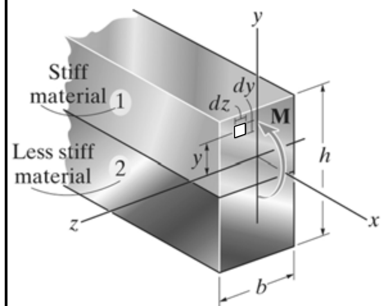
Bending-stress variation for beam transformed to material 2

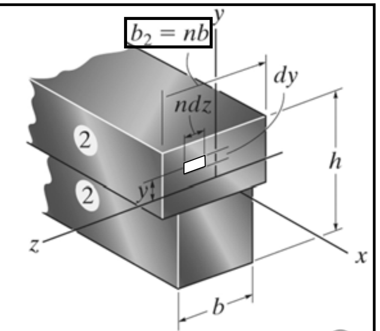


Bending-stress variation for beam transformed to material 1

✓ Once the beam has been transformed into one having a single material, the normal-stress distribution over the transformed cross section will be linear.

✓ The centroid (neutral axis) and moment of inertia for the transformed area can be determined and the flexure formula applied in the usual manner to determine the stress at each point on the transformed beam.





Beam transformed to material 2

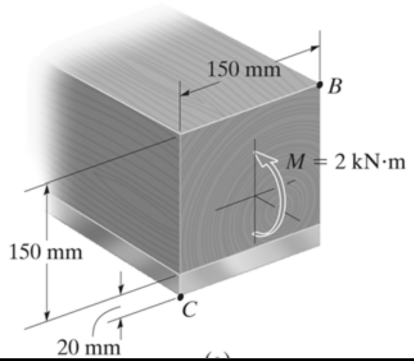
$$dF = \sigma dA = \sigma dz dy \qquad dF' = \sigma' dA' = \sigma' n dz dy$$

Equivalent force $dF = dF'$

$$\sigma dz dy = \sigma' n dz dy \quad \Rightarrow \quad \boxed{\sigma = n\sigma'}$$

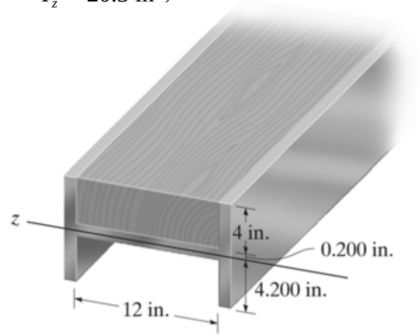
EXAMPLE 6.17

A composite beam is made of wood and reinforced with a steel strap located on its bottom side. It has the cross-sectional area shown in Figure. If the beam is subjected to a bending moment of $M = 2\ \text{kN}\cdot\text{m}$, determine the normal stress at the point B and C. Take $E_w = 12$ GPa and $E_{st} = 200$ GPa .

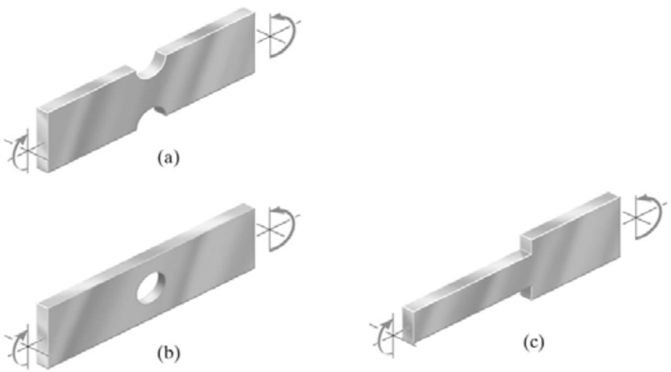


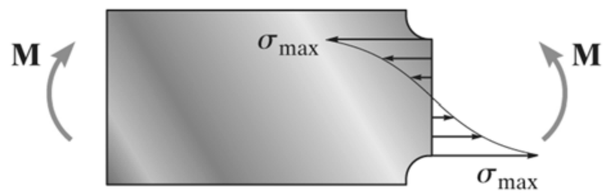
EXAMPLE 6.**

In order to reinforce the steel beam, an oak board is placed between its flanges as shown in Figure. If the allowable normal stress for the steel is $(\sigma_{allow})_{st} = 24$ ksi , and for the wood $(\sigma_{allow})_{w} = 3$ ksi , determine the maximum bending moment the beam can support, with and without the wood reinforcement. $E_{st} = 29(10^3)$ ksi, $E_w = 1.6(10^3)$ ksi. The moment of inertia of the steel beam is $I_z = 20.3\ \text{in}^4$, and its cross-sectional area is $A = 8.79\ \text{in}^2$.



*6.9 Stress Concentrations

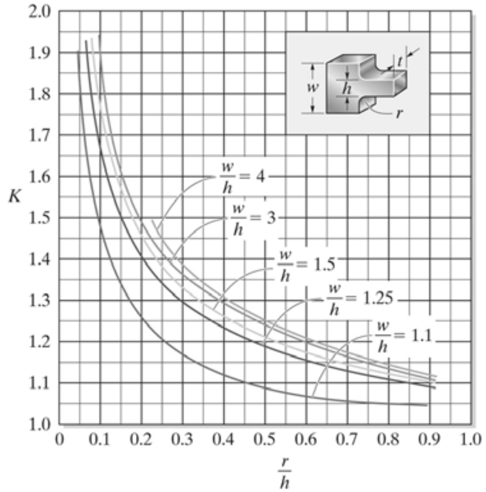


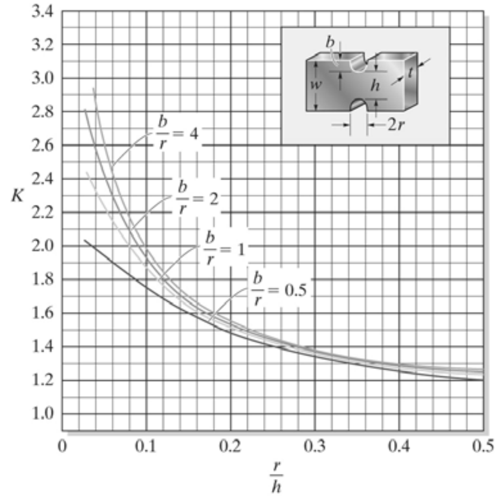


✓ The normal-stress and strain distributions at the section become nonlinear and can be obtained only through experiment or, in some case, by a mathematical analysis using the theory of elasticity.

$$\sigma_{max} = K\frac{Mc}{I}$$

K : stress-concentration factor



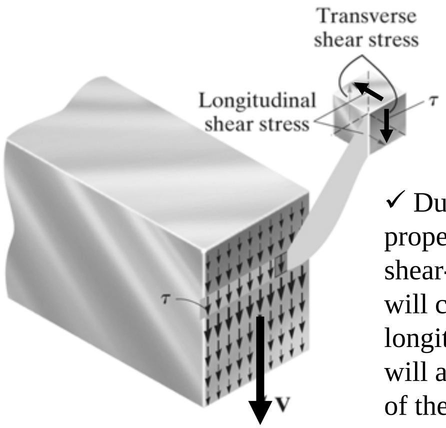


Stress concentrations caused by bending occur at the sharp corners of this window lintel and are responsible for the cracks at the corners.

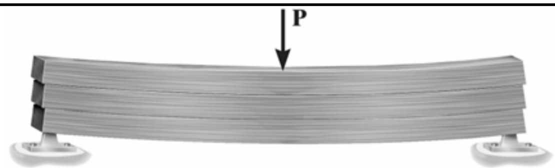
Chapter 7
Transverse Shear

Railroad ties act as beams that support very large transverse shear loadings. As a result, if they are made of wood, they will tend to split at their ends, where the shear loads are the largest.

7.1 Shear in Straight Members

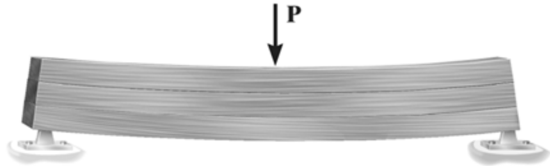


✓ Due to the complementary property of shear, the transverse shear-stress on the cross section will create corresponding longitudinal shear stresses which will act along longitudinal planes of the beam.



Boards not bonded together

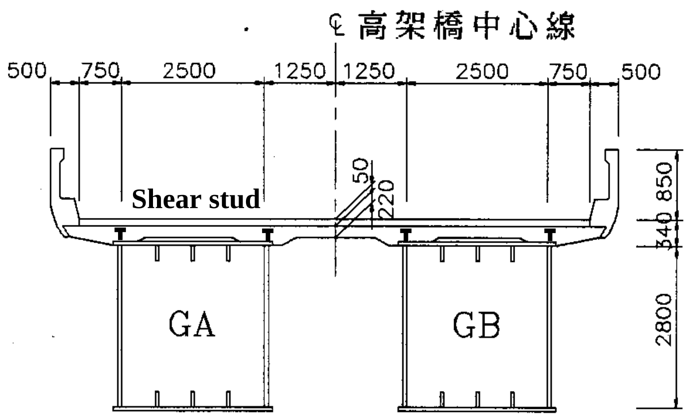
✓ The load **P** causes the boards to slide relative to one another, and so the beam deflects.

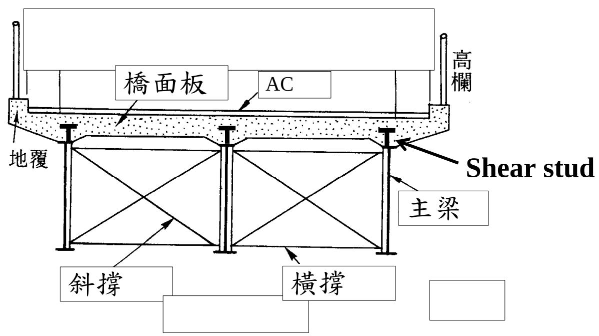


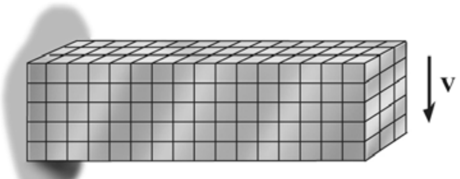
Boards bonded together

✓ The longitudinal shear stresses acting between the boards will prevent the relative sliding, and consequently the beam will act as a single unit.

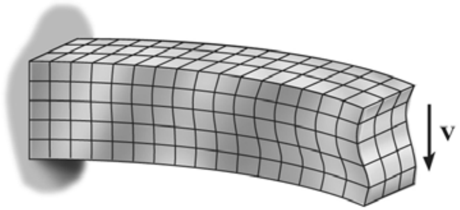
Shear connectors are “tack welded” to this corrugated metal floor liner so that when the concrete floor is poured, the connectors will prevent the concrete slab from slipping on the liner surface. The two materials will thus act as a composite slab.







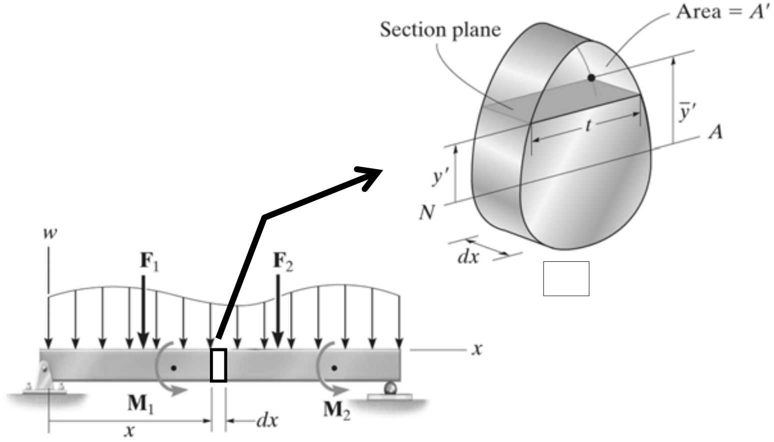
(a) Before deformation

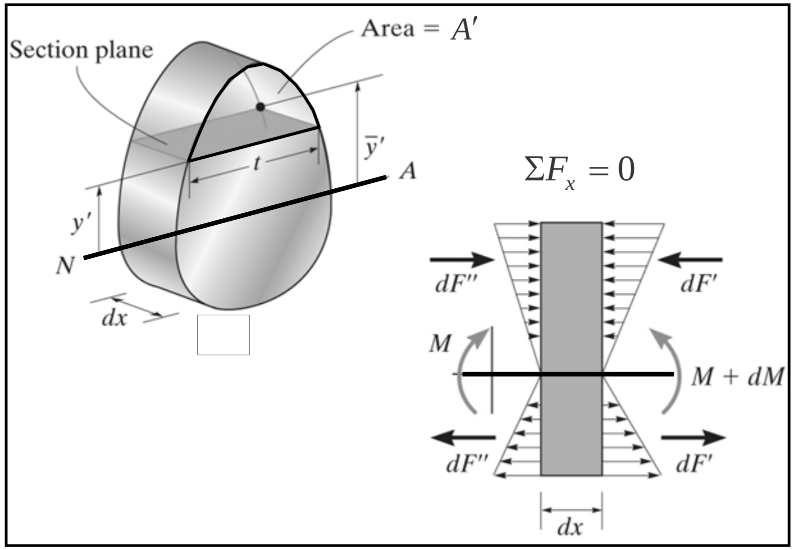


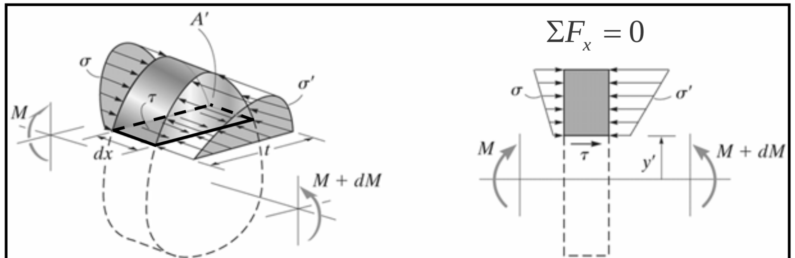
(b) After deformation

- ✓ As a result of the shear stress, shear strains will be developed and these will tend to distort the cross section in a rather complex manner.
- ✓ This nonuniform shear-strain distribution will cause the cross section to warp.
- ✓ Assume the cross-sectional warping is small enough so that it can be neglected. i.e. cross sections remain plane.
- ✓ This assumption is particularly true for the most common case of a slender beam which has a small depth compared with its length.

7.2 The Shear Formula





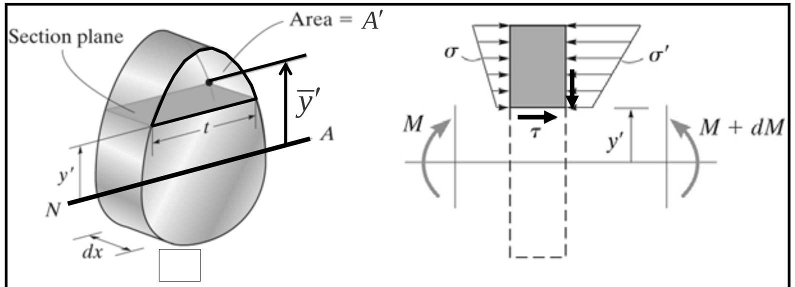


✓ Assume the shear stress is uniformly distributed over the width ***t*** at the section where the shear stress is determined.

$$(+\leftarrow)\ \Sigma F_x = 0 \qquad \int_{A'} \sigma' dA' - \int_{A'} \sigma dA' - \tau(t\ dx) = 0$$

$$\int_{A'}\left(\frac{M+dM}{I}\right) y dA' - \int_{A'}\left(\frac{M}{I}\right) y dA' - \tau(t\ dx) = 0$$

$$\left(\frac{dM}{I}\right)\int_{A'} y\ dA' = \tau(t\ dx) \quad \longrightarrow \quad \tau = \frac{1}{I\ t}\left(\frac{dM}{dx}\right)\int_{A'} y\ dA'$$

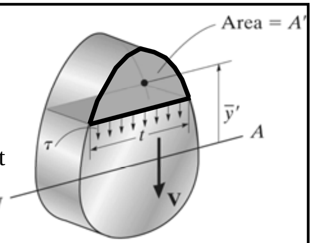


$$\frac{dM}{dx} = V \qquad Q = \int_{A'} y\, dA' = \bar{y}' A' \qquad \bar{y}' = \frac{\int_{A'} y\, dA'}{A'}$$

$$\tau = \frac{1}{I\,t}\left(\frac{dM}{dx}\right)\int_{A'} y\, dA' \quad \Rightarrow \quad \boxed{\tau = \frac{VQ}{I\,t}}$$

$$\boxed{\tau = \frac{VQ}{I\,t}}$$

Shear formula



τ : the shear stress in the member at the point located a distance y' from the neutral axis

V : the internal resultant shear force

I : the moment of inertia of the *entire* cross-sectional area computed about the neutral axis

t : the width of the member's cross-sectional area, measured at the point where τ is to be determined

Q : $\int_{A'} y\, dA' = \bar{y}' A'$, where A' is the top (or bottom) portion of the member's cross-sectional area, defined from the section where t is measured, and $\bar{y}'$ is the distance to the centroid of A', measured from the neutral axis.

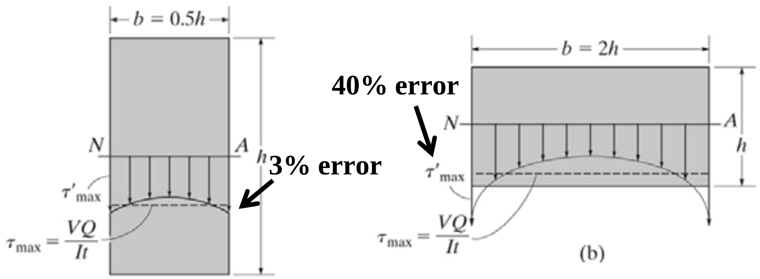
✓ It is necessary that the material behave in a linear elastic manner and have a modulus of elasticity that is the same in tension as it is in compression.

○ Limitations on the Use of the Shear Formula

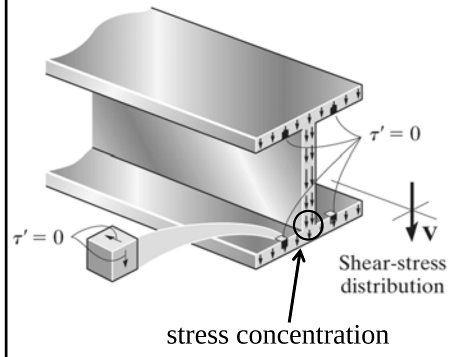
✓ One of the major assumptions in the shear formula is that the shear stress is uniformly distributed over the width ***t*** at the section where the shear stress is determined.

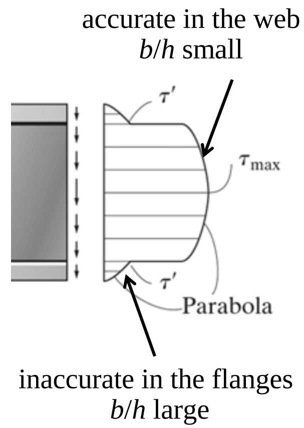
✓ The maximum value, τ'_{max} , occurs at the edges of the cross section, and its magnitude depends on the ratio ***b/h*** (width/depth).

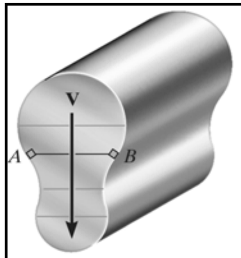
✓ As the b/h ratio increases, the error increases.



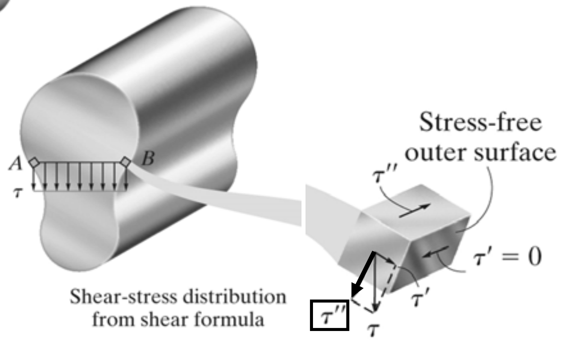
Wide-Flange Beam





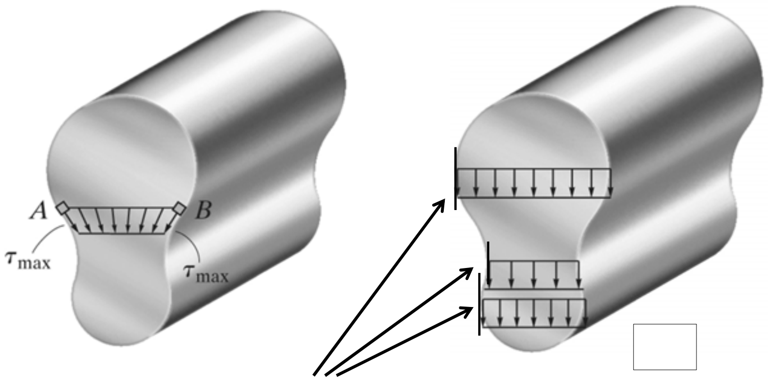


A beam having a cross section with an irregular or nonrectangular boundary



tangent to the boundary

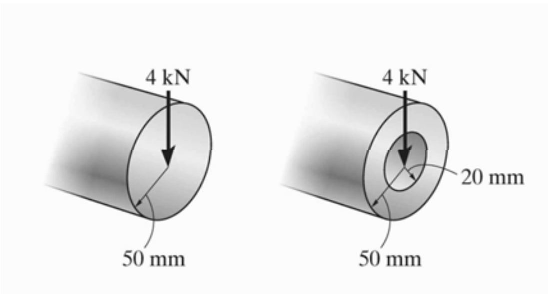
Shear formula cannot be applied.



Shear formula can be applied.

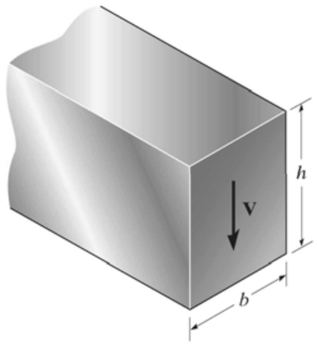
EXAMPLE 7.1

The solid shaft and tube shown in Figure are subjected to the shear force of 4 kN. Determine the shear stress acting over the diameter of each cross section.

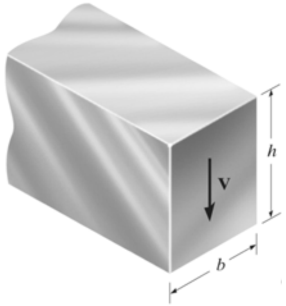


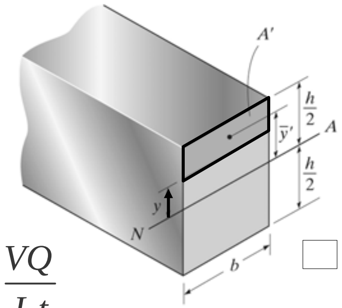
EXAMPLE 7.2

Determine the distribution of the shear stress over the cross section of the beam shown in Figure.



○ Rectangular Cross Section





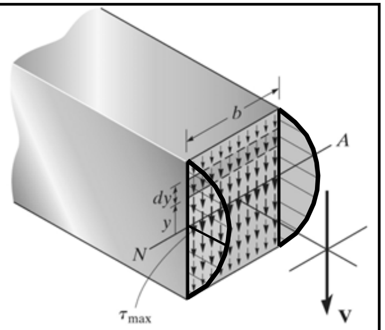
$$\tau = \frac{VQ}{I\ t}$$

$$Q = \bar{y}'A' = \left[y + \frac{1}{2}\left(\frac{h}{2} - y\right)\right]\left(\frac{h}{2} - y\right)b = \frac{1}{2}\left(\frac{h^2}{4} - y^2\right)b$$

Applying the shear formula

$$\tau = \frac{VQ}{I\ t} = \frac{V\left(\frac{1}{2}\right)\left[\frac{h^2}{4} - y^2\right]b}{\left(\frac{1}{12}bh^3\right)b}$$

$$\tau = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right)$$



Shear–stress distribution

At the neutral axis, $y = 0,\ A = bh$

$$\tau_{\max} = 1.5\frac{V}{A} = 1.5\ \tau_{avg}$$

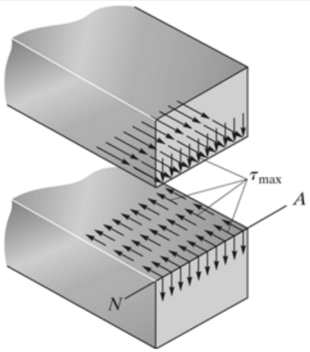
At $y = \pm h/2$

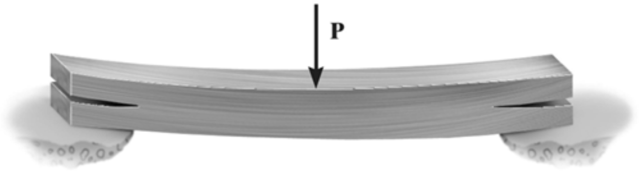
$$\tau_{\min} = 0$$

The same value can be obtained directly from the shear formula.

$$\tau_{\max} = \frac{VQ}{I\ t} = \frac{V(h/4)(bh/2)}{\left[\frac{1}{12}bh^3\right]b}$$

$$= 1.5\frac{V}{A} = 1.5\ \tau_{avg}$$





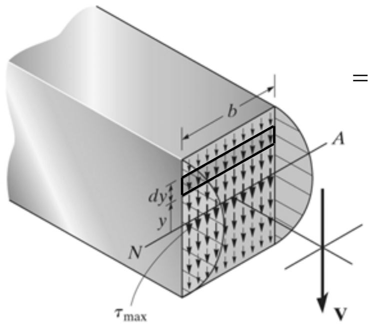
$$\tau = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right)$$

$$dA = b\ dy$$

$$\int_A \tau\ dA = \int_{-h/2}^{h/2} \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right)bdy$$

$$= \frac{6V}{h^3}\left[\frac{h^2}{4}y - \frac{1}{3}y^3\right]_{-h/2}^{h/2}$$

$$= \frac{6V}{h^3}\left[\frac{h^2}{4}\left(\frac{h}{2} + \frac{h}{2}\right) - \frac{1}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right)\right] = V$$



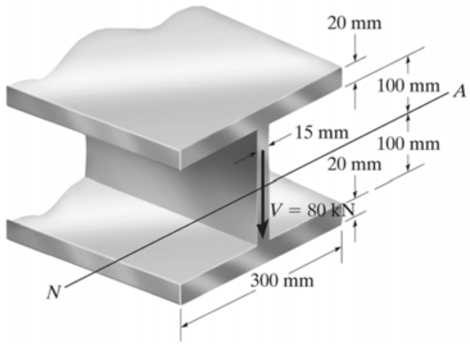
Shear–stress distribution

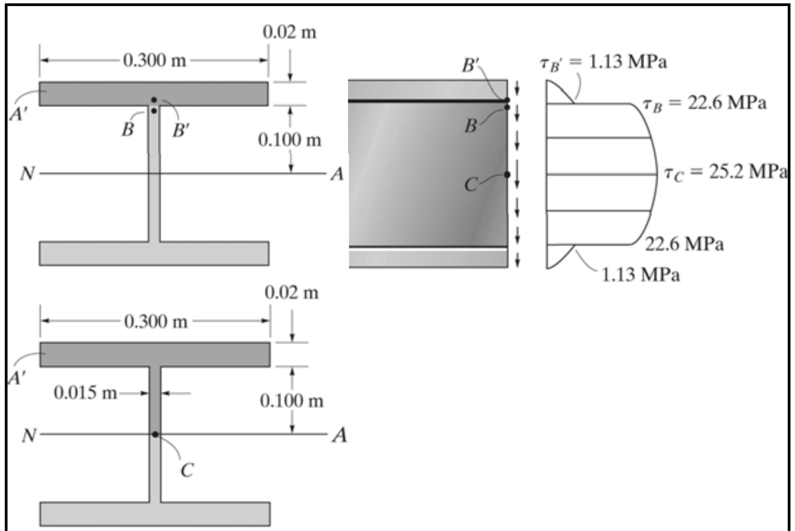
✓ When the shear-stress distribution is integrated over the cross section, it yields the resultant shear *V*.

Typical shear failure of this wooden beam occurred at the support and through the approximate center of its cross section.

EXAMPLE 7.3

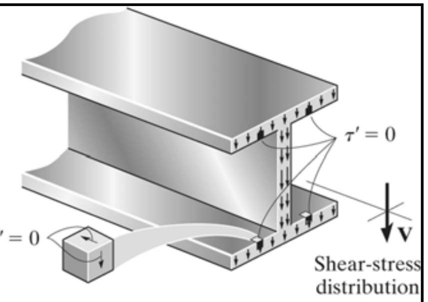
A steel wide-flange beam has the dimensions shown in Figure. If it is subjected to a shear of V = 80 kN, plot the shear-stress distribution acting over the beam's cross-sectional area.

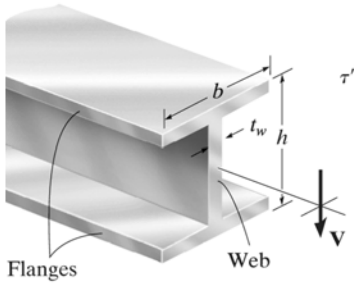


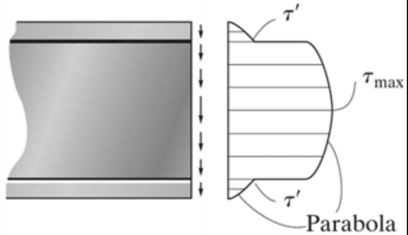


○ Wide-Flange Beam

✓ A wide-flange beam consists of two (wide) "flanges" and a "web".



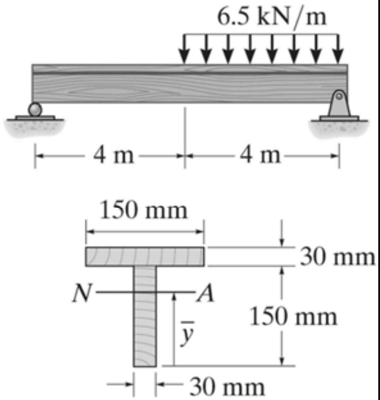




✓ The web will carry significantly more of the shear force than the flanges.

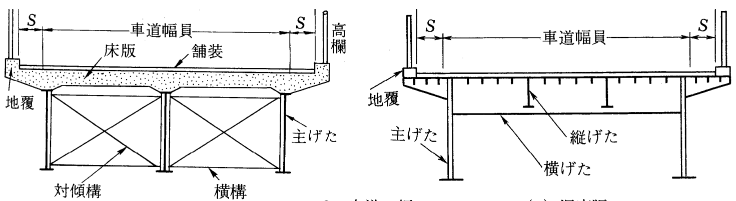
EXAMPLE 7.4

The beam shown in Figure is made from two boards. Determine the maximum shear stress in the glue necessary to hold the boards together along the seam where they are joined.

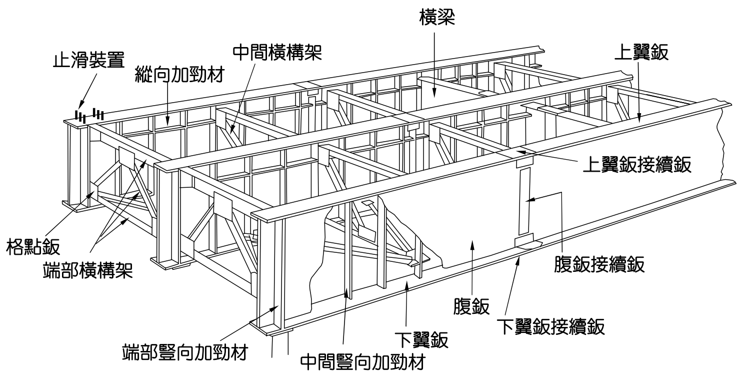


7.3 Shear Flow in Built-Up Members

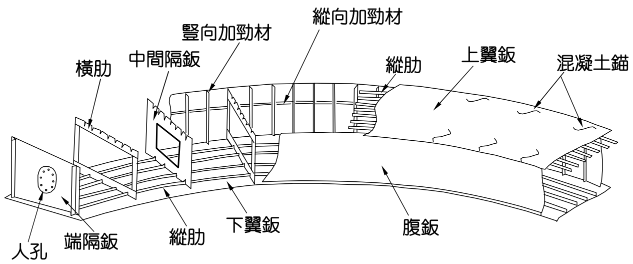
✓ Members are "built up" from several composite parts in order to achieve a greater resistance to loads.
✓ Fasteners such as nails, bolts, welding material, or glue may be needed to keep the component parts from sliding relative to one another.
✓ To design fasteners, it is necessary to know the shear force that must be resisted by the fastener along the member's length.
✓ ***Shear flow q*** is a measure of ***the force per unit length*** along a ***longitudinal axis*** of a beam.



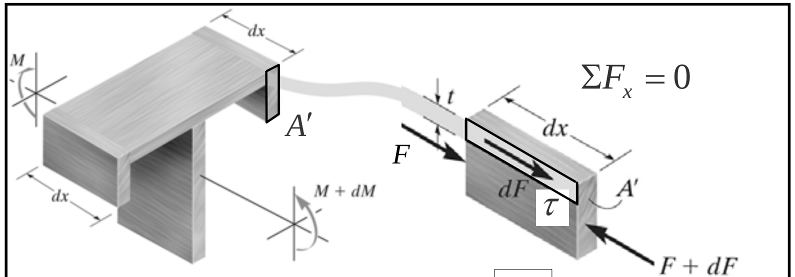
（a）RC 床板 　 S：歩道の幅 　（b）鋼床版



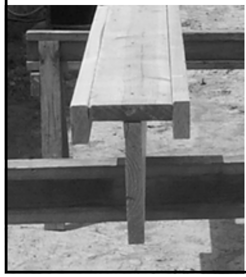
鋼鈑梁橋之組成



曲線鋼箱型梁橋之組成



$$\Sigma F_x = 0$$



$$dF = \left(\frac{dM}{I}\right)\int_{A'} y\,dA' = \left(\frac{dM}{I}\right)Q = \tau(t\ dx)$$

Shear flow $q = \dfrac{dF}{dx} = \left(\dfrac{dM}{dx}\right)\dfrac{Q}{I} = \dfrac{VQ}{I} = \tau\ t$

$$\Rightarrow \boxed{q = \frac{VQ}{I}}$$

$$\boxed{q = \frac{VQ}{I}}$$ Shear flow formula

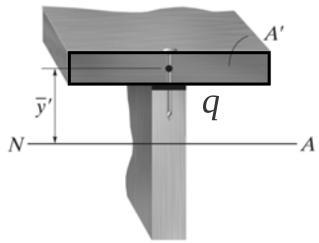
q : the shear flow, measured as a force per unit length along the beam

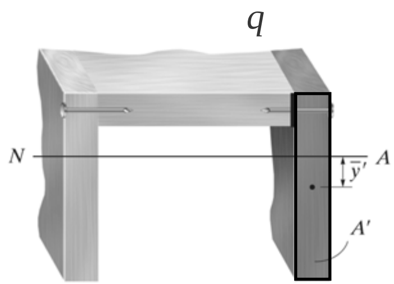
V : the internal resultant shear force

I : the moment of inertia of the *entire* cross-sectional area computed about the neutral axis

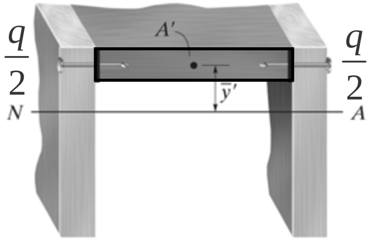
Q : $\int_{A'} y\ dA' = \bar{y}' A'$, where A' is the cross-sectional area of the segment that is connected to the beam at the juncture where the shear flow is to be calculated, any $\bar{y}'$ is the distance from the neutral axis to the centroid of A'.

Single fastener

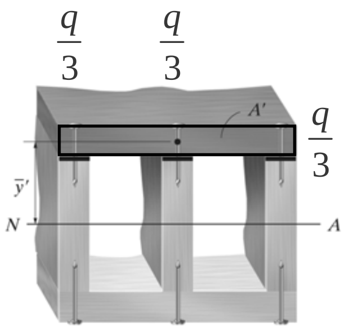




Two fasteners

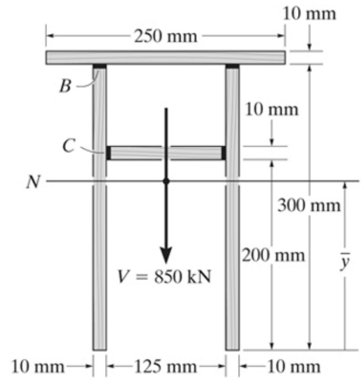


Three fasteners



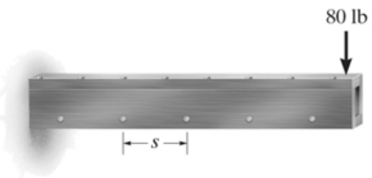
EXAMPLE 7.5

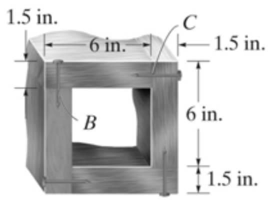
The beam is constructed from four boards glued together as shown in Figure. If it is subjected to a shear of V=850 kN, determine the shear flow at B and C that must be resisted by the glue.



EXAMPLE 7.6

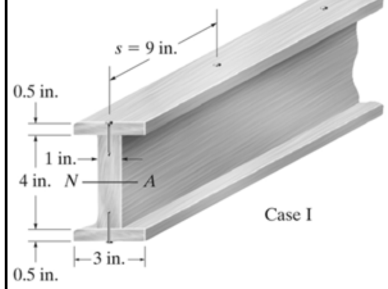
A box beam is constructed from four boards nailed together as shown in Figure. If each nail can support a shear force of 30 lb, determine the maximum spacing ***s*** of nails at B and at C so that the beam will support the force of 80 lb.

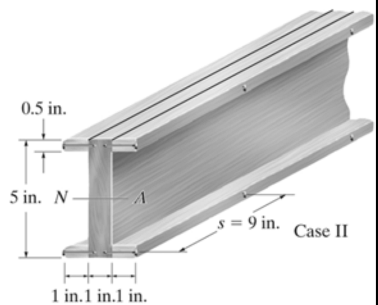




EXAMPLE 7.7

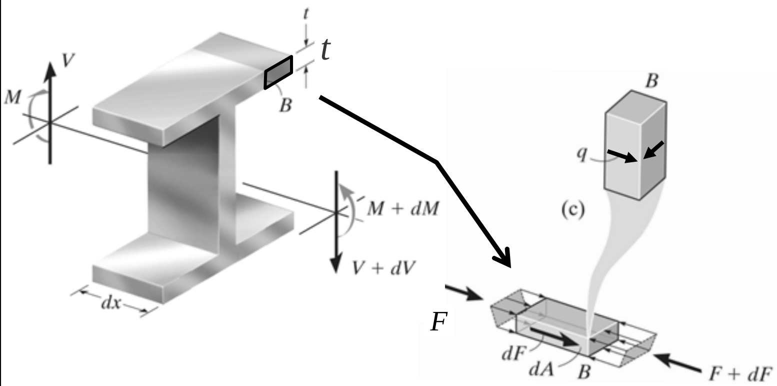
Nails having a total shear strength of 40 lb are used in a beam that can be constructed either as in Case I or as in Case II, in Figure. If the nails are spaced at 9 in., determine the largest vertical shear that can be supported in each case so that the fasteners will not fail.



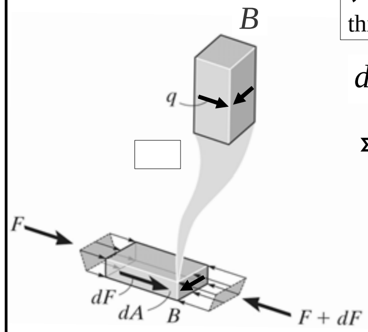


7.4 Shear Flow in Thin-Walled Members

➢ Thin-walled members: the wall thickness is small compared with the height or width of the member.



✓ The shear stress τ will not vary much over the thickness ***t*** of the section.



Shear flow $q = \dfrac{dF}{dx} = \left(\dfrac{dM}{dx}\right)\dfrac{Q}{I} = \dfrac{VQ}{I}$

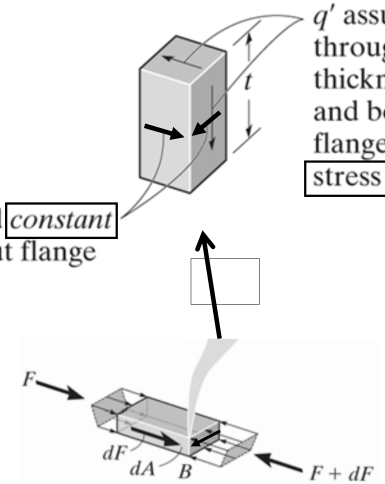
✓ The shear stress τ is constant over the thickness ***t*** of the section.

$$dF = \tau dA = \tau(t\ dx) = qdx$$

$$\Rightarrow \quad \boxed{q = \tau t}$$

$$\Uparrow$$

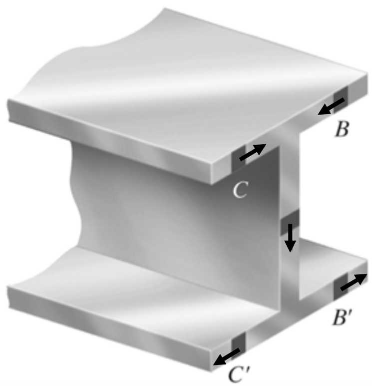
From $q = \dfrac{VQ}{I}$ and $\tau = \dfrac{VQ}{I\ t}$

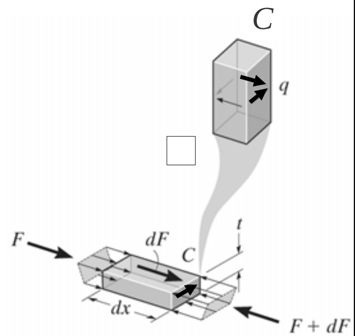


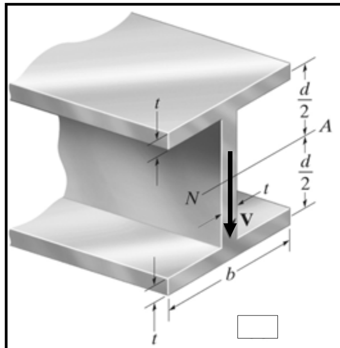
q' assumed to be zero throughout flange thickness since top and bottom of flange are stress free

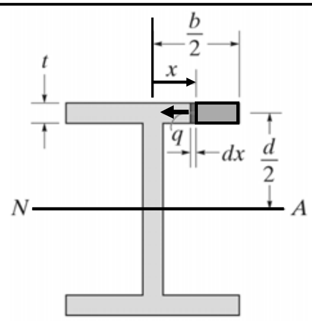
q assumed *constant* throughout flange thickness

Direction of the shear flow







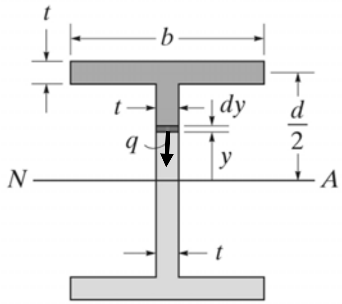


✓ The shear force **V** must act along an axis of symmetry or principal centroidal axis of inertia for the cross section.

$$Q = \bar{y}'A' = [d/2]((b/2)-x)t$$

$$q = \frac{VQ}{I} = \frac{V[d/2]((b/2)-x)t}{I}$$

$$= \frac{V\,t\,d}{2I}\left(\frac{b}{2}-x\right) \quad \text{linear}$$



$$Q = \Sigma\,\bar{y}'A'$$

$$= \frac{d}{2}(bt) + \left[y + \frac{1}{2}\left(\frac{d}{2}-y\right)\right]t\left(\frac{d}{2}-y\right)$$

$$= \frac{btd}{2} + \frac{1}{2}t\left(\frac{d^2}{4}-y^2\right)$$

$$q = \frac{VQ}{I} = \frac{Vt}{I}\left[\frac{db}{2} + \frac{1}{2}\left(\frac{d^2}{4}-y^2\right)\right] \quad \text{parabolic}$$

Flange

$$q = \frac{V\,t\,d}{2I}\left(\frac{b}{2}-x\right) \quad \text{linear}$$

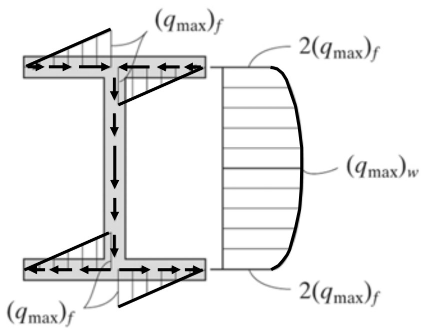
$$(q_{\max})_f = \frac{V\,t\,db}{4I}$$

Web

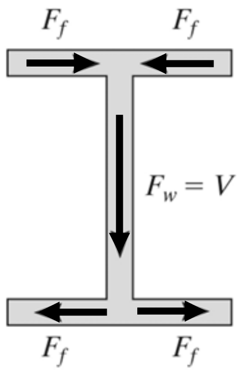
$$q = \frac{Vt}{I}\left[\frac{db}{2} + \frac{1}{2}\left(\frac{d^2}{4}-y^2\right)\right]$$

parabolic

$$(q_{\max})_w = \frac{Vt}{I}\left(\frac{db}{2} + \frac{d^2}{8}\right)$$



Shear-flow distribution



$$F_f = \int q\,dx = \int_0^{b/2} \frac{Vtd}{2I}\left(\frac{b}{2}-x\right)dx$$

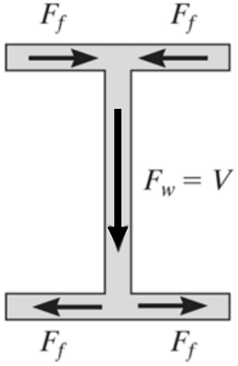
$$= \frac{V\,t\,db^2}{16I}$$

$$\text{or} \quad F_f = \frac{1}{2}(q_{\max})_f\left(\frac{b}{2}\right) = \frac{V\,t\,db^2}{16I}$$

$$F_w = \int q\,dy = \int_{-d/2}^{d/2} \frac{Vt}{I}\left[\frac{db}{2} + \frac{1}{2}\left(\frac{d^2}{4}-y^2\right)\right]dy$$

$$= \frac{Vt}{I}\left[\frac{db}{2}y + \frac{1}{2}\left(\frac{d^2}{4}y - \frac{1}{3}y^3\right)\right]\Bigg|_{-d/2}^{d/2}$$

$$= \frac{V\,t\,d^2}{4I}\left(2b + \frac{1}{3}d\right)$$



$$F_w = \frac{V\,t\,d^2}{4I}\left(2b + \frac{1}{3}d\right)$$

$$I = 2\left[\frac{1}{12}bt^3 + bt\left(\frac{d}{2}\right)^2\right] + \frac{1}{12}td^3$$

Neglecting the first term, since the thickness of each flange is small, we get

$$I = \frac{td^2}{4}\left(2b + \frac{1}{3}d\right)$$

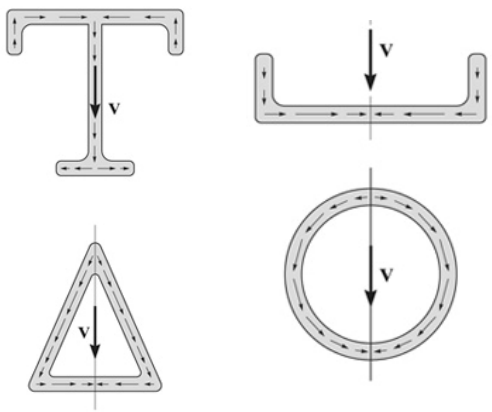
$$\Rightarrow \quad F_w = V$$

Important Remarks

(1) The value of ***q*** changes over the cross section, since Q will be different for each area segment A' for which it is determined. In particular, ***q*** will vary *linearly* along segments (flanges) that are *perpendicular* to the direction of **V**, and *parabolically* along segments (web) that are *inclined or parallel* to **V**.

(2) ***q*** will always act parallel to the walls of the member, since the section on which ***q*** is calculated is taken perpendicular to the walls.

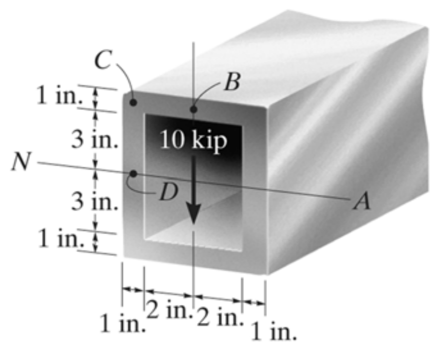
(3) The *directional sense* of ***q*** is such that the shear appears to "*flow*" through the cross section, inward at the beam's top flange, "combining" and then "flowing" *downward* through the web, since it must contribute to the shear force **V**, and then separating and "flowing" *outward* at the bottom flange.



✓ Symmetry prevails about an axis that is collinear with **V**, and as a result, ***q*** "flows" in a direction such that it will provide the necessary vertical force components equivalent to **V** and yet also satisfy the horizontal force equilibrium requirements for the cross section.

EXAMPLE 7.8

The thin-walled box beam in figure is subjected to a shear of 10 kip. Determine the variation of the shear flow throughout the cross section.



Chapter 8
Combined Loadings

The offset column supporting this sign is subjected to the combined loadings of normal force, shear force, bending moment and torsion.

The offset hanger supporting this ski gondola is subjected to the combined loadings of axial force and bending moment.

8.1 Thin-Walled Pressure Vessels

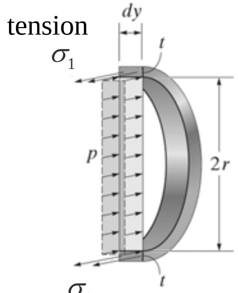
- "Thin wall" refers to a vessel having an inner-radius-to-wall-thickness ratio of 10 or more $(r/t \geq 10)$.
- When $r/t = 10$, the results of a thin-wall analysis will predict a stress that is approximately 4% less than the actual maximum stress in the vessel.
- For larger r/t this error will be even smaller.
- The stress distribution throughout the thickness of thin-walled pressure vessels will not vary significantly, so we will assume that it is *uniform or constant*.
- The pressure in the vessel is understood to be the *gauge pressure*, since it measures the pressure *above* atmosphere pressure, which is assumed to exist both inside and outside the vessel's wall.

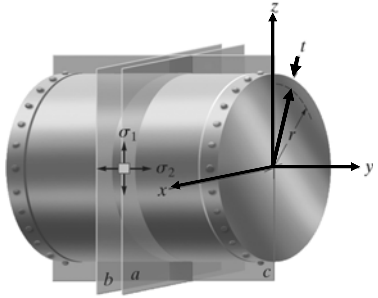
Cylindrical pressure vessels, such as this gas tank, have semi-spherical end caps rather than flat ones in order to reduce the stress in the tank.

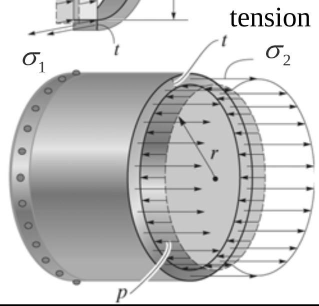
○ Cylindrical Vessels

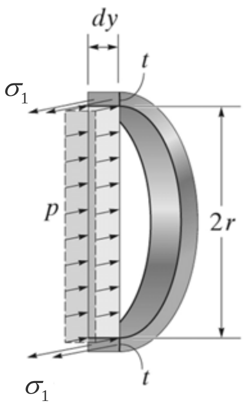
✓ Normal stresses σ_1 in the ***circumferential or hoop*** direction

✓ Normal stresses σ_2 in the ***longitudinal or axial*** direction









✓ *circumferential or hoop stress*

$$\Sigma F_x = 0$$

$$2\left[\sigma_1(t\,dy)\right] - p(2r\,dy) = 0$$

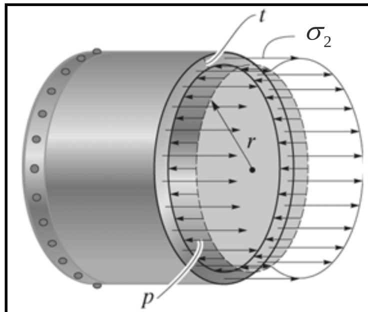
$$\Rightarrow \boxed{\sigma_1 = \frac{pr}{t}}$$

σ_1 : the normal stress in hoop direction, assumed to be constant throughout the wall of the cylinder, and subjects the material to tension

p : the internal gauge pressure developed by the contained gas or fluid

r : the inner radius of the cylinder

t : the thickness of the wall $(r/t \geq 10)$



✓ *longitudinal or axial stress*

$$\Sigma F_y = 0$$

$$\sigma_2(2\pi rt) - p(\pi r^2) = 0$$

$$\Rightarrow \boxed{\sigma_2 = \frac{pr}{2t}}$$

σ_2 : the normal stress in longitudinal direction, assumed to be constant throughout the wall of the cylinder, and subjects the material to tension

p : the internal gauge pressure developed by the contained gas or fluid

r : the inner radius of the cylinder

t : the thickness of the wall $(r/t \geq 10)$

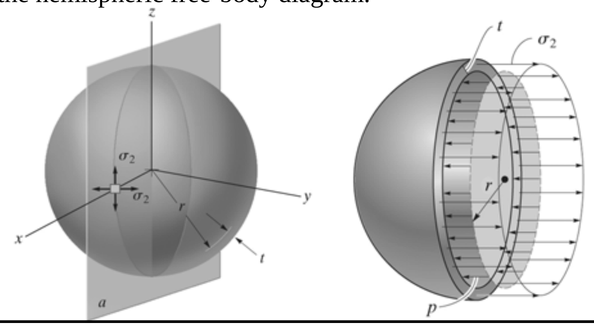
$\sigma_1 = 2\sigma_2$ ***hoop stress > longitudinal stress***

Shown is the barrel of a shotgun which was clogged with debris just before firing. Gas pressure from the charge increased the circumferential stress within the barrel enough to cause the rupture.

○ **Spherical Vessels**

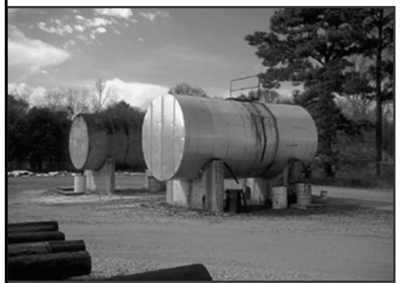
$$\Sigma F_y = 0 \qquad \sigma_2(2\pi r t) - p(\pi r^2) = 0 \quad \Rightarrow \quad \boxed{\sigma_2 = \frac{pr}{2t}}$$

✓ the same result as that obtained for the longitudinal stress in the cylindrical pressure vessel
✓ the stress is the same regardless of the orientation of the hemispheric free-body diagram.



EXAMPLE 8.1

A cylindrical pressure vessel has an inner diameter of 4 ft and a thickness of ½ in. Determine the maximum internal pressure it can sustain so that neither its circumferential nor its longitudinal stress component exceeds 20 ksi. Under the same conditions, what is the maximum internal pressure that a similar-size spherical vessel can sustain?



8.2 State of Stress Caused by Combined Loadings

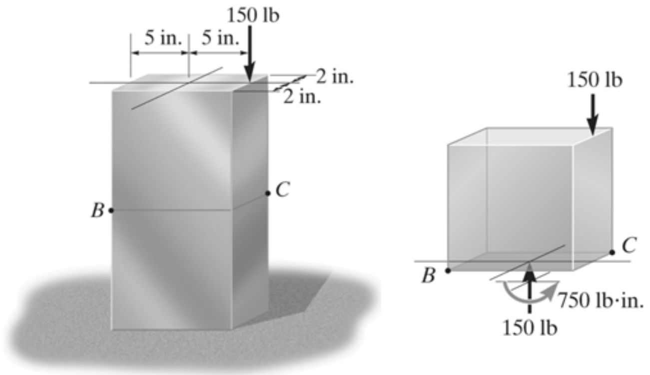
➢ Most often, the cross section of a member is subjected to several of the types of loadings simultaneously, an internal axial force, a shear force, a bending moment and a torsional moment.

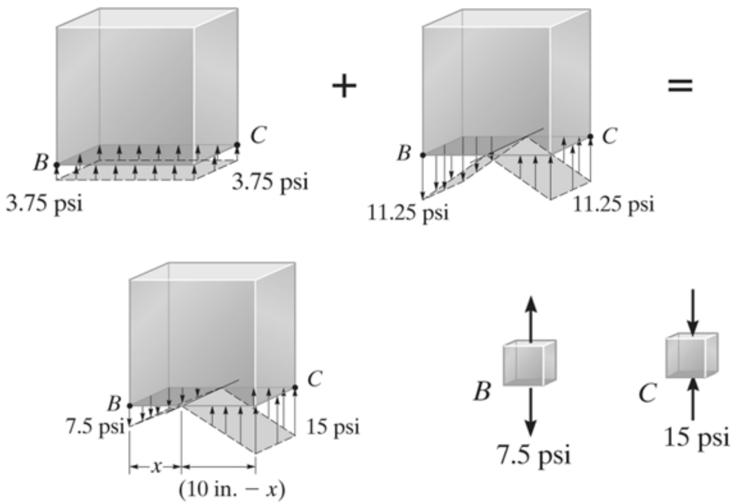
➢ The method of superposition can be used to determine the resultant stress distribution caused by the loads, provided a *linear relationship* exists between the stress and the loads and the geometry of the member should *not* undergo *significant change* when the loads are applied.

This chimney is subjected to the combined loading of wind and weight. It is important to investigate the tensile stress in the chimney since masonry is weak in tension.

EXAMPLE 8.2

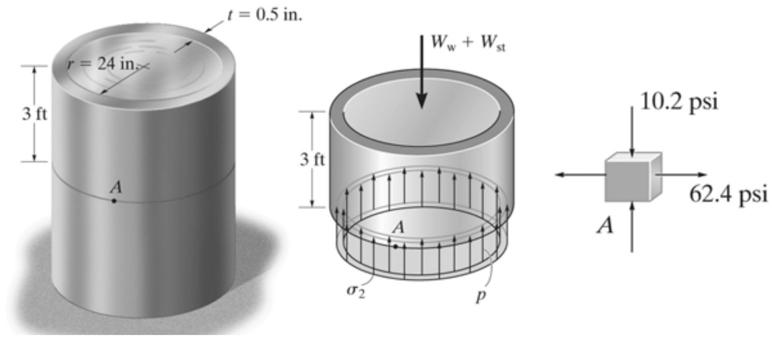
A force of 150 lb is applied to the edge of the member shown in Figure. Neglect the weight of the member and determine the state of stress at points *B* and *C*.





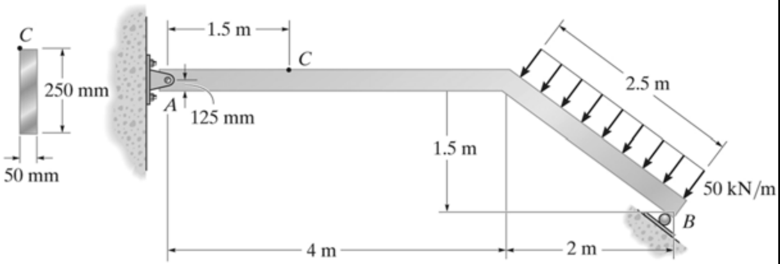
EXAMPLE 8.3

The tank in Figure has an inner radius of 24 in. and a thickness of 0.5 in. It is filled to the top with water having a specific weight of $\gamma_w = 62.4\ \text{lb/ft}^3$. If it is made of steel having a specific weight of $\gamma_{st} = 490\ \text{lb/ft}^3$, determine the state of stress at point *A*. The tank is open at the top.



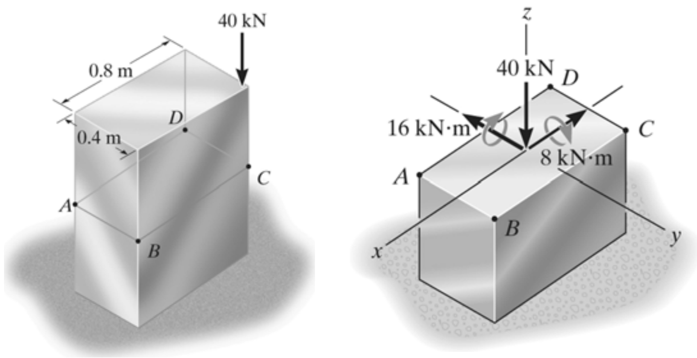
EXAMPLE 8.4

The member shown in Figure has a rectangular cross section. Determine the state of stress that the loading produces at point *C*.



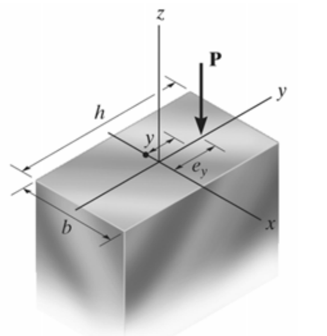
EXAMPLE 8.5

The rectangular block of negligible weight in Figure is subjected to a vertical force of 40 kN, which is applied to its corner. Determine the normal-stress distribution acting on a section through *ABCD*.



EXAMPLE 8.6

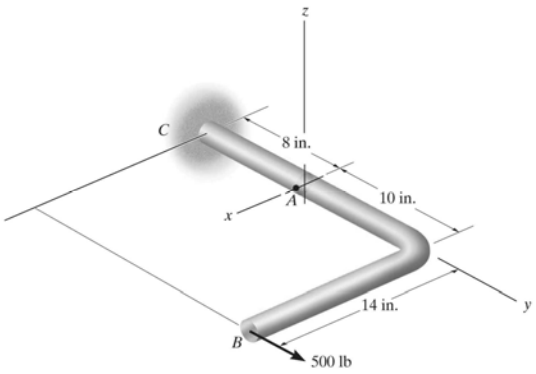
A rectangular block has a negligible weight and is subjected to a vertical force **P**, in Figure. (a) Determine the range of values for the eccentricity e_y of the load along the *y* axis so that it does not cause any tensile stress in the block. (b) Specify the region on the cross section where **P** may be applied without causing a tensile stress in the block.



Here is an example of where combined axial and bending stress can occur.

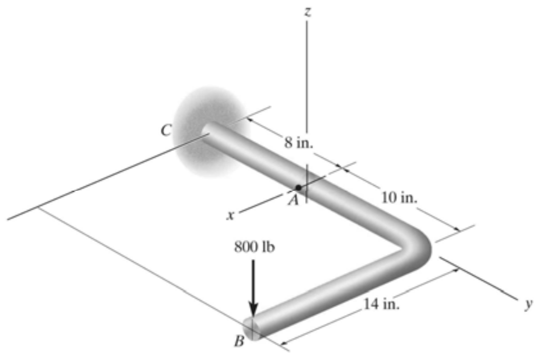
EXAMPLE 8.7

The solid rod shown in Figure has a radius of 0.75 in. If it is subjected to the force of 500 lb, determine the state of stress at point *A*.



EXAMPLE 8.8

The solid rod shown in Figure has a radius of 0.75 in. If it is subjected to the force of 800 lb, determine the state of stress at point *A*.

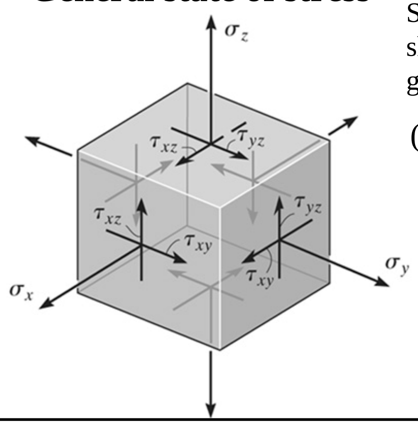


Chapter 9
Stress Transformation

These turbine blades are subjected to a complex pattern of stress. For design it is necessary to determine where and in what direction the maximum stress occurs.

9.1 Plane-Stress Transformation

General state of stress

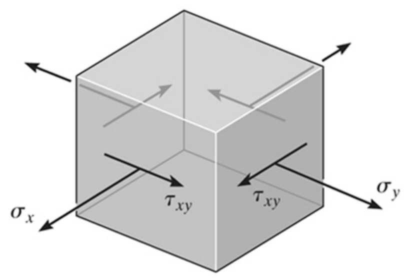


Six independent normal and shear stress components in a general stress state

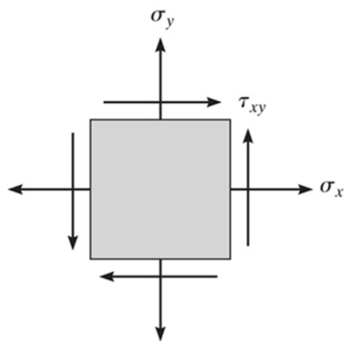
$$(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$$

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy}$$
$$\tau_{zx} = \tau_{xz}$$

Plane stress

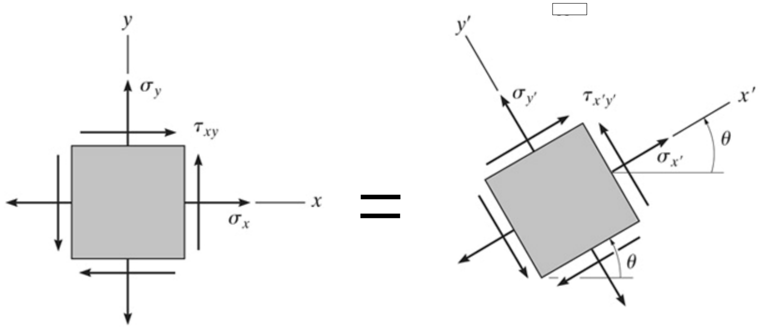


(three-dimensional view)

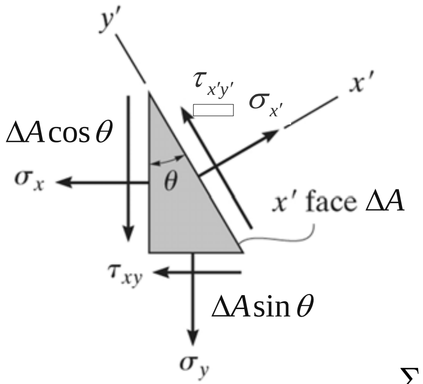


(two-dimensional view)

✓ Engineers frequently make approximations or simplifications of the loadings on a body in order that the stress produced in a structural member or mechanical element can be analyzed in a ***single plane***. When this is the case, the material is said to be subjected to ***plane stress***.

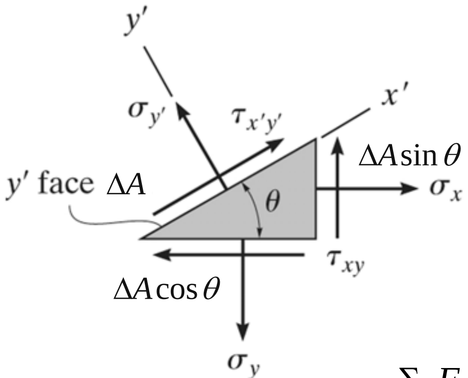


✓ **The state of plane stress at the point is uniquely represented by three components acting on an element that has a specific orientation at the point.**



$$\Sigma\ F_{x'} = 0 \longrightarrow \sigma_{x'}$$

$$\Sigma\ F_{y'} = 0 \longrightarrow \tau_{x'y'}$$

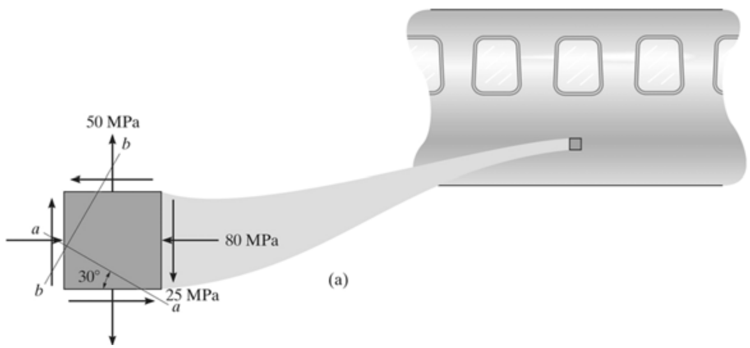


$$\Sigma\ F_{y'} = 0 \longrightarrow \sigma_{y'}$$

$$\Sigma\ F_{x'} = 0 \longrightarrow \tau_{x'y'}$$

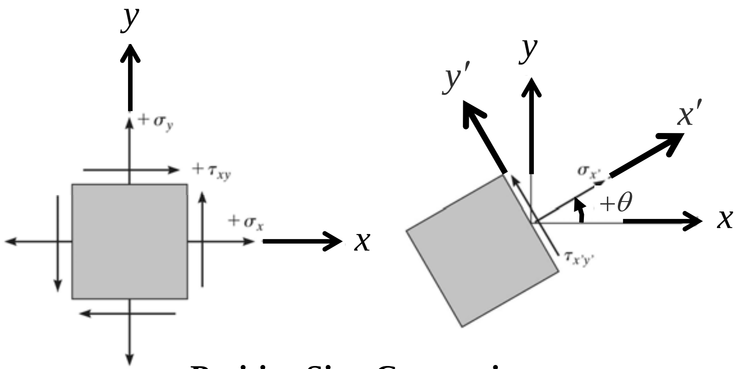
EXAMPLE 9.1

The state of plane at a point on the surface of the airplane fuselage is represented on the element oriented as shown in figure. Represent the state of stress at the point on an element that is oriented 30 ° clockwise from the position shown.



(a)

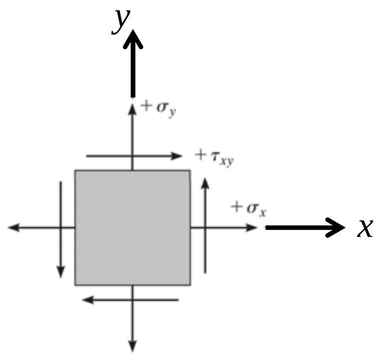
9.2 General Equations of Plane-Stress Transformation



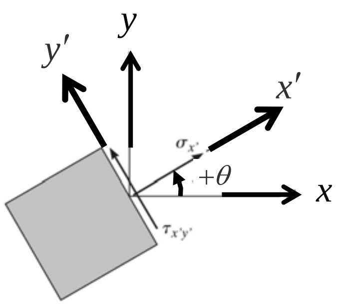
Positive Sign Convention

○ Sign Convention

A normal or shear stress component is positive provided it acts in the ***positive*** coordinate direction on the ***positive*** face of the element, or it acts in the ***negative*** coordinate direction on the ***negative*** face of the element.



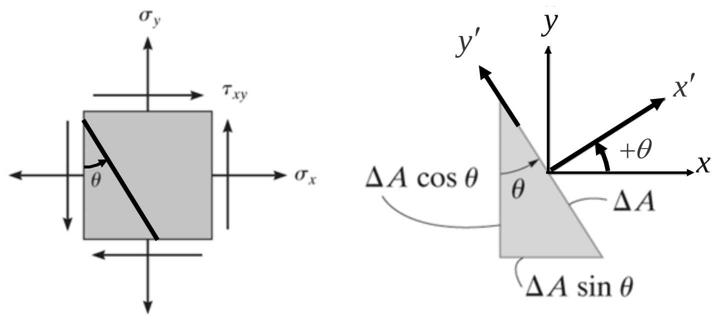
✓ Positive normal stress acts outward from all faces and positive shear stress acts upward on the right-hand face of the element.

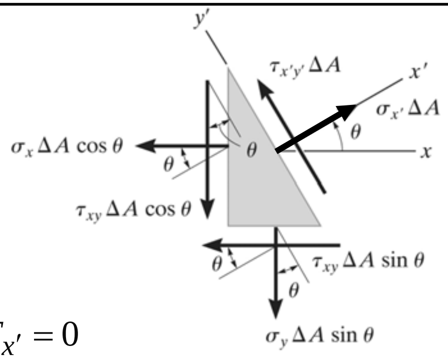


✓ Right-handed coordinate systems

✓ The angle θ is measured from the positive *x* to the positive *x'* axis.

○ Normal and Shear Stress Components

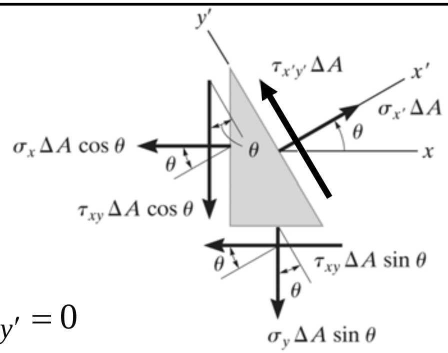




$$\Sigma F_{x'} = 0$$

$$\sigma_{x'}\Delta A - \left(\tau_{xy}\Delta A\sin\theta\right)\cos\theta - \left(\sigma_y\Delta A\sin\theta\right)\sin\theta$$
$$-\left(\tau_{xy}\Delta A\cos\theta\right)\sin\theta - \left(\sigma_x\Delta A\cos\theta\right)\cos\theta = 0$$

$$\Rightarrow \sigma_{x'} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau_{xy}\left(2\sin\theta\cos\theta\right)$$



$$\Sigma F_{y'} = 0$$

$$\tau_{x'y'}\Delta A + \left(\tau_{xy}\Delta A\sin\theta\right)\sin\theta - \left(\sigma_y\Delta A\sin\theta\right)\cos\theta$$
$$-\left(\tau_{xy}\Delta A\cos\theta\right)\cos\theta + \left(\sigma_x\Delta A\cos\theta\right)\sin\theta = 0$$

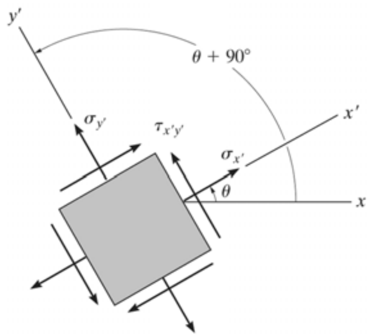
$$\Rightarrow \tau_{x'y'} = \left(\sigma_y - \sigma_x\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\sigma_{x'} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau_{xy}\left(2\sin\theta\cos\theta\right)$$
$$\tau_{x'y'} = \left(\sigma_y - \sigma_x\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

By using $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin^2\theta = \left(1-\cos 2\theta\right)/2$ $\quad$ $\cos^2\theta = \left(1+\cos 2\theta\right)/2$

$$\Rightarrow \boxed{\begin{aligned} \sigma_{x'} &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{x'y'} &= -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{aligned}}$$

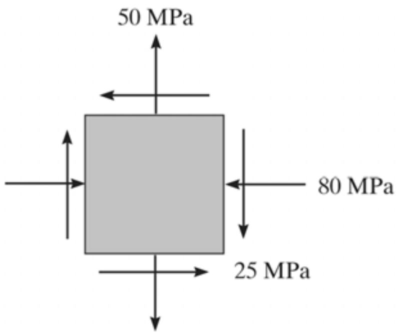


By substituting $\theta = \theta + 90°$

$$\Rightarrow \boxed{\sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta}$$

EXAMPLE 9.2

The state of plane stress at a point is represented by the element shown in Figure. Determine the state of stress at the point on another element oriented 30 °clockwise from the position shown.



9.3 Principal Stresses and Maximum In-Plane Shear Stress

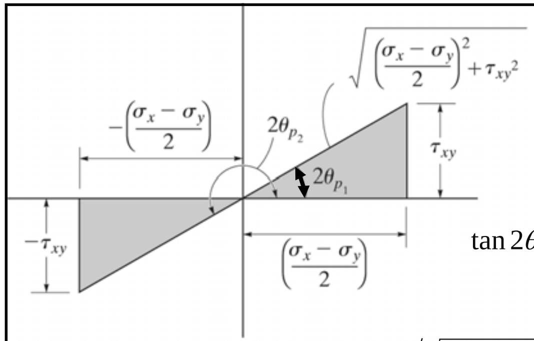
○ In-Plane Principal Stresses

To determine the maximum and minimum normal stress, we must differentiate $\sigma_{x'}$ with respect to θ and set the result equal to zero.

$$\frac{d\sigma_{x'}}{d\theta} = -\frac{\sigma_x - \sigma_y}{2}\left(2\sin 2\theta\right) + 2\tau_{xy}\cos 2\theta = 0$$

the orientation $\theta = \theta_p$ of the planes of maximum and minimum normal stress

$$\boxed{\tan 2\theta_p = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2}} \rightarrow \theta_{p1} \text{ and } \theta_{p2}$$

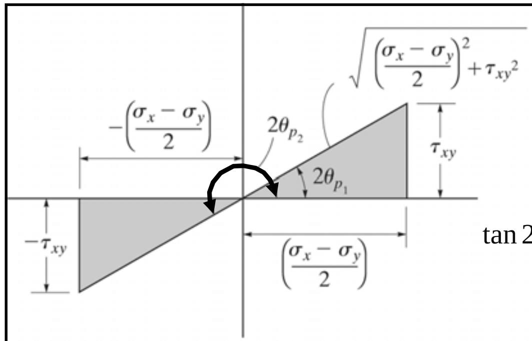


$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2}$$

For θ_{p1}

$$\sin 2\theta_{p1} = \tau_{xy} \Big/ \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\cos 2\theta_{p1} = \left(\frac{\sigma_x - \sigma_y}{2}\right) \Big/ \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

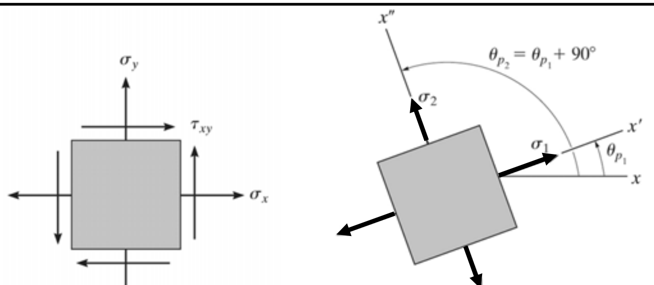


$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2}$$

For θ_{p2}

$$\sin 2\theta_{p2} = -\tau_{xy} \Big/ \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\cos 2\theta_{p2} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \Big/ \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



In-plane principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

This particular set of values are called the in-plane ***principal stresses***, and the corresponding planes on which they act are called the ***principal planes*** of stress.

For θ_{p1}

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

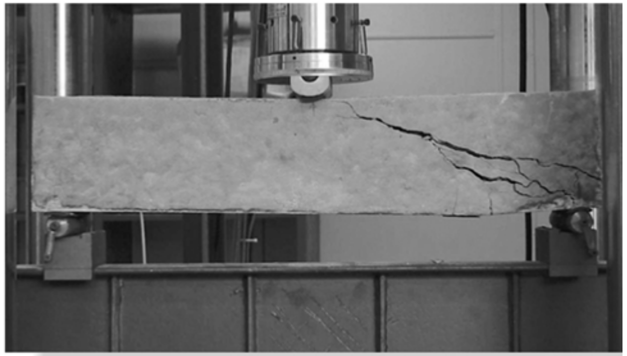
$$= -\frac{\sigma_x - \sigma_y}{2}\tau_{xy} \Big/ \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$+\tau_{xy}\left(\frac{\sigma_x - \sigma_y}{2}\right) \Big/ \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$= 0$$

For θ_{p2} $\quad \tau_{x'y'} = 0$

✓ **No shear stress acts on the principal planes**



The cracks in this concrete beam were caused by tension stress, even though the beam was subjected to both an internal moment and shear. The stress-transformation equations can be used to predict the direction of the cracks, and the principal normal stresses that caused them.

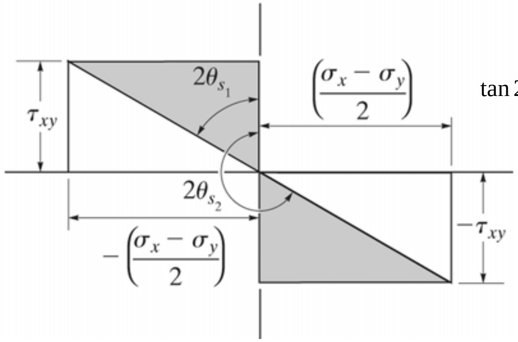
○ Maximum In-Plane Shear Stress

The orientation of an element that is subjected to maximum shear stress on its faces can be determined by taking the derivative of $\tau_{x'y'}$ with respect to θ and setting the result equal to zero.

$$\frac{d\tau_{x'y'}}{d\theta} = 0$$

The orientation $\theta = \theta_s$ of the planes of maximum shear stress

$$\boxed{\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}}} \Rightarrow \theta_{s1} \text{ and } \theta_{s2}$$



$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

✓ The planes for maximum shear stress can be determined by orienting an element 45 ° from the position of an element that defines the planes of principal stress.

Maximum In-Plane Shear Stress

$$\boxed{\tau_{\substack{\max \\ in-plane}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}}$$

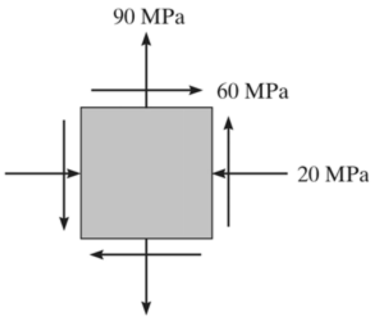
A normal stress on the planes of maximum in-plane shear stress

$$\boxed{\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}}$$

EXAMPLE 9.3

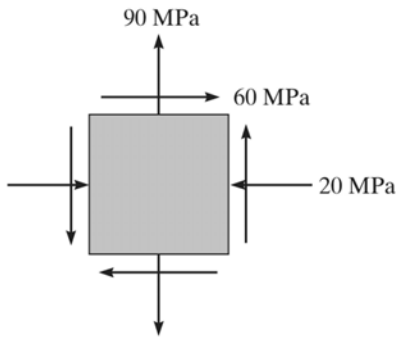
The state of plane stress at a failure point on the shaft is shown on the element in Figure. Represent this stress state in terms of the principal stresses.

Notice how the failure plane is at an angle due to tearing of the material.



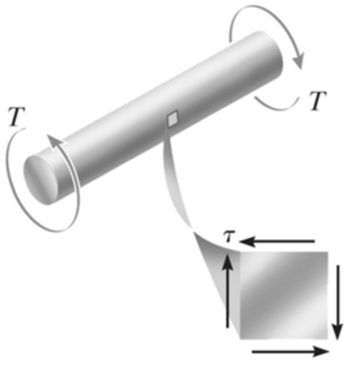
EXAMPLE 9.4

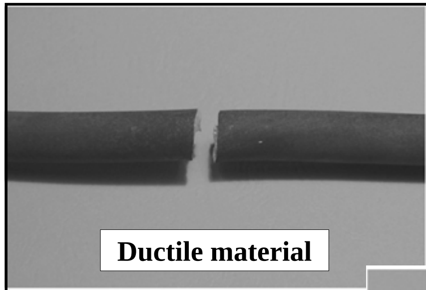
The state of plane stress at a point on a body is represented on the element shown in Figure. Represent this stress in terms of the maximum in-plane shear stress and associated average normal stress.

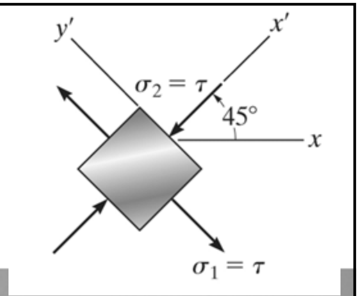


EXAMPLE 9.5

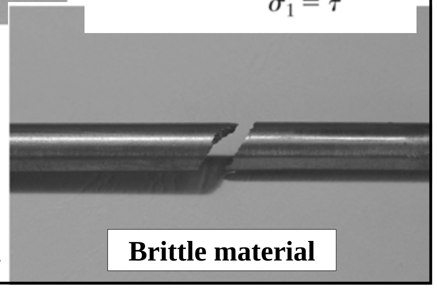
When the torsional loading T is applied to the bar in Figure. It produces a state of pure shear stress in the material. Determine (a) the maximum in-plane shear stress and the associated average normal stress, and (b) the principal stress.







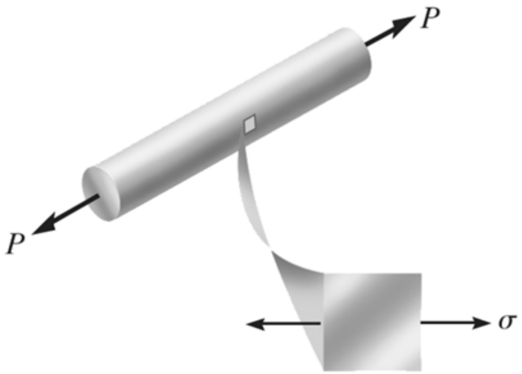
Fail due to shear stress

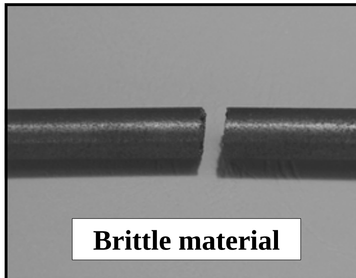


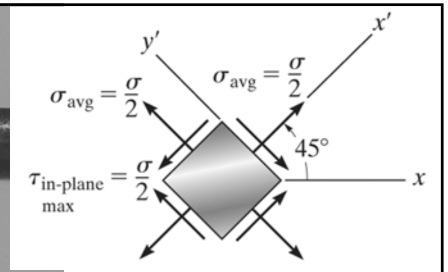
Fail due to tensile stress

EXAMPLE 9.6

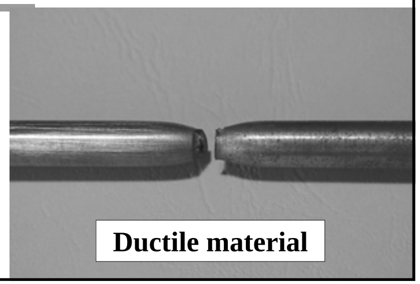
When the axial loading **P** is applied to the bar in Figure, it produces a tensile stress in the material. Determine (a) the principal stress and the (b) maximum in-plane shear stress and associated average normal stress.





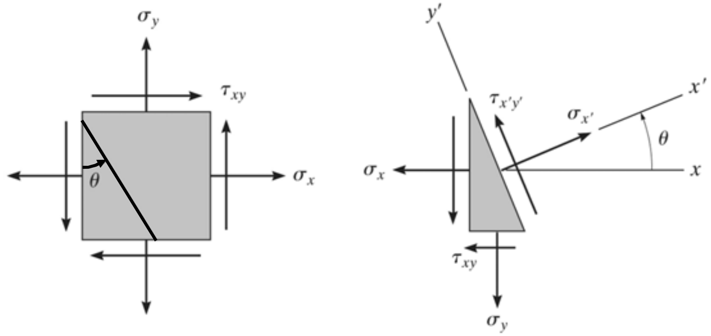


Fail due to normal stress



Fail due to shear stress

9.4 Mohr's Circle - Plane Stress



$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

The parameter θ can be eliminated by squaring each equation and adding the equations together.

$$\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

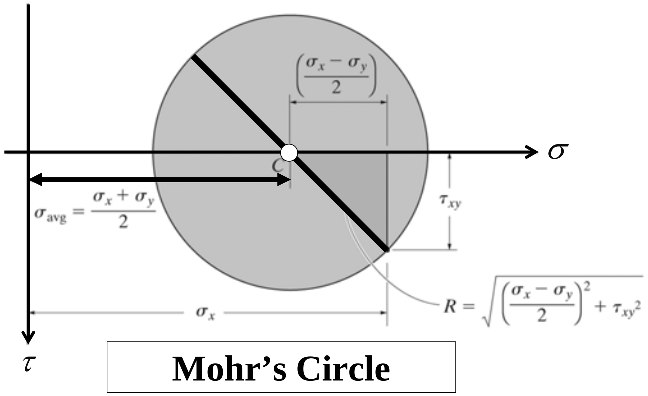
$$\Rightarrow \boxed{\left(\sigma_{x'} - \sigma_{avg}\right)^2 + \tau_{x'y'}^2 = R^2}$$

A circle having a radius R and center on the σ axis at point $(\sigma_{avg}, 0)$

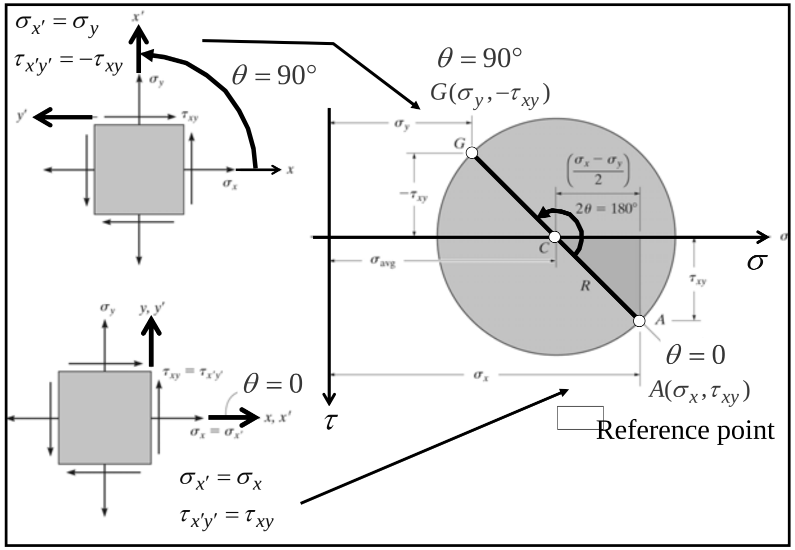
where $\sigma_{avg} = \dfrac{\sigma_x + \sigma_y}{2}$

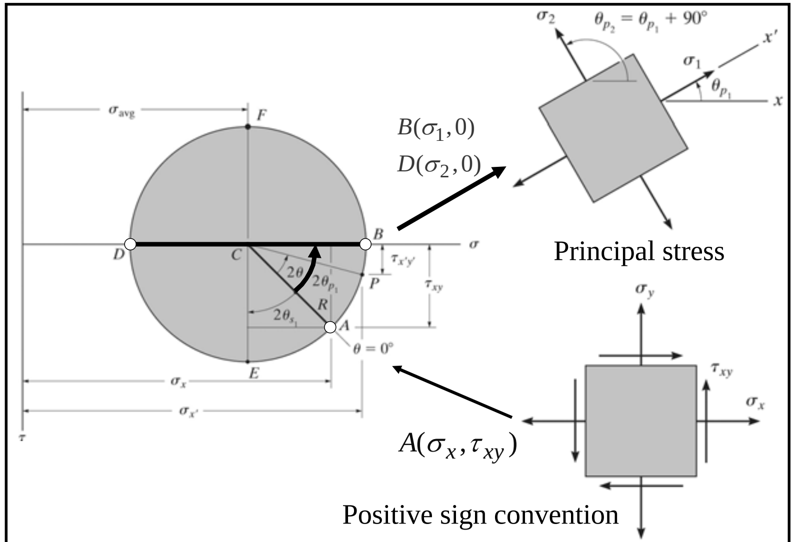
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

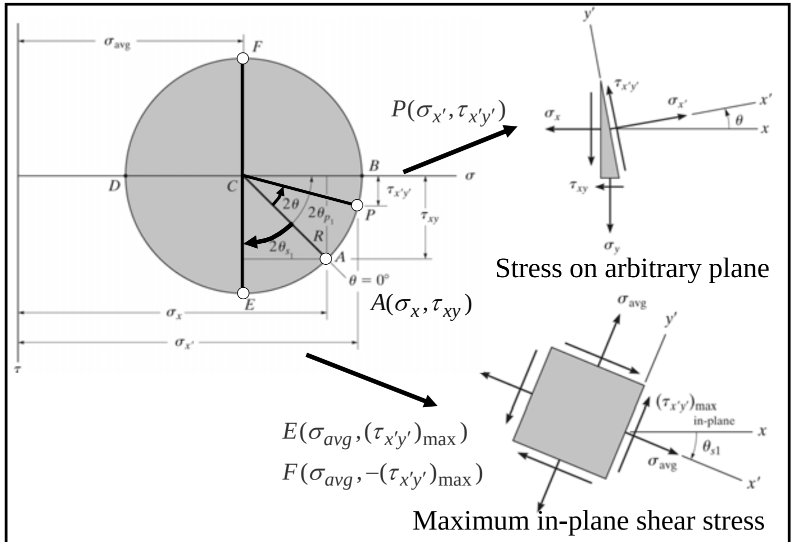
$$\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$



Mohr's Circle

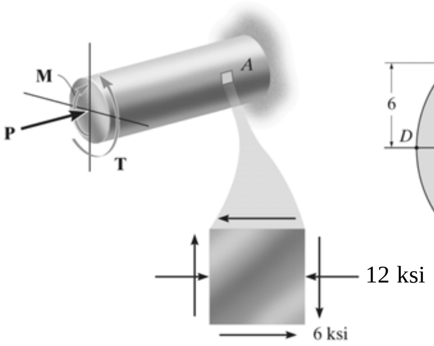


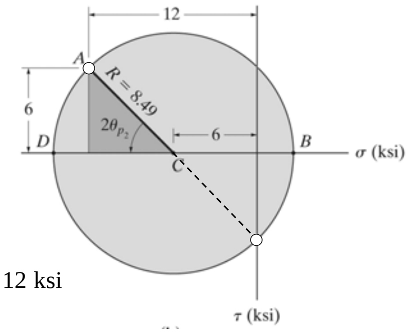




EXAMPLE 9.7

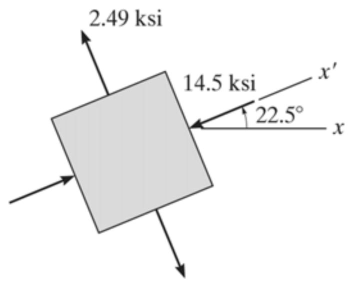
Due to the applied loading, the element at point A on the solid shaft in Figure is subjected to the state of stress shown. Determine the principal stresses acting at this point.

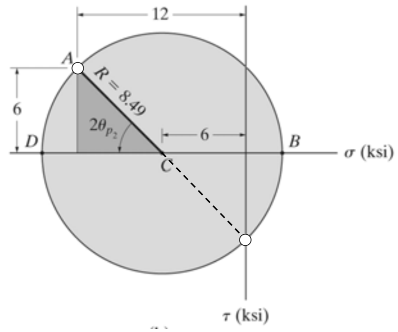




EXAMPLE 9.7

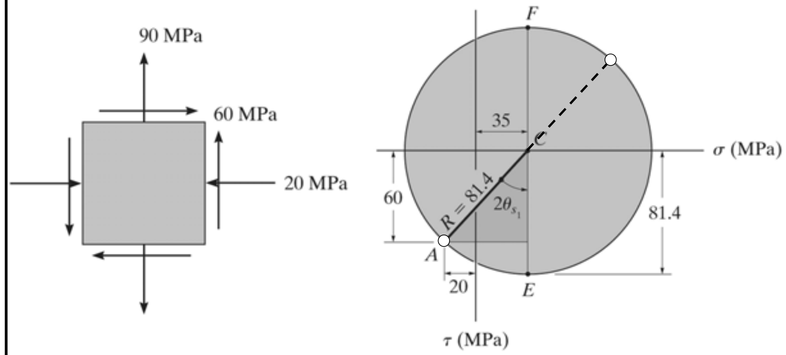
Due to the applied loading, the element at point A on the solid shaft in Figure is subjected to the state of stress shown. Determine the principal stresses acting at this point.





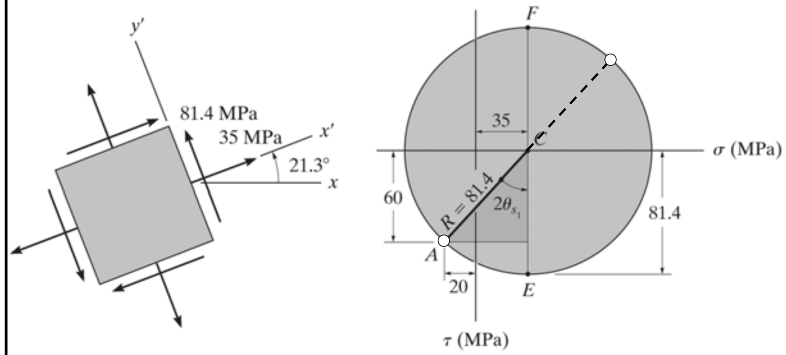
EXAMPLE 9.8

The state of plane stress at a point is shown on the element in Figure. Determine the maximum in-plane shear stresses at this point.



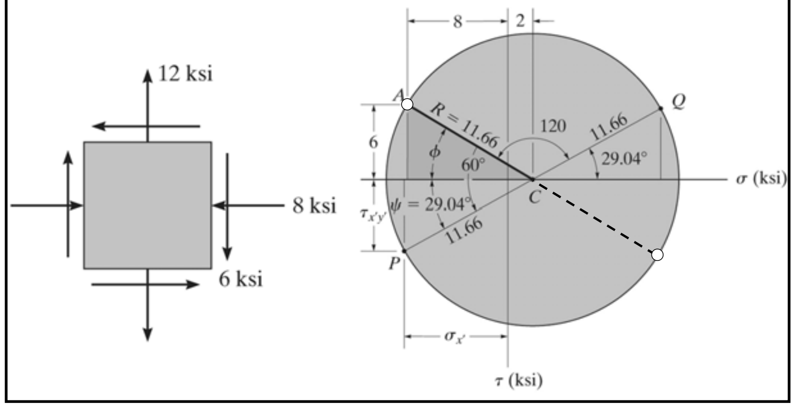
EXAMPLE 9.8

The state of plane stress at a point is shown on the element in Figure. Determine the maximum in-plane shear stresses at this point.



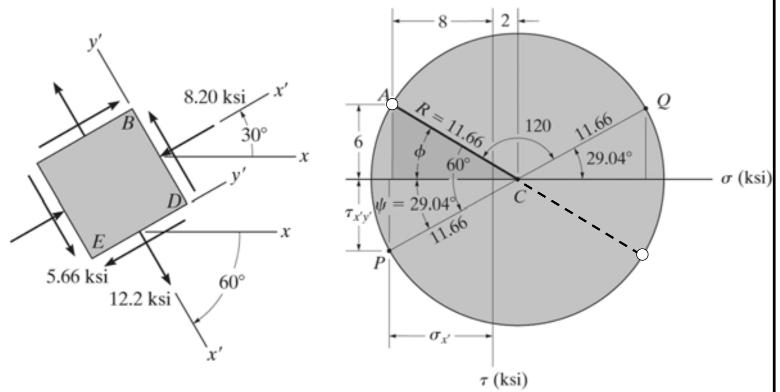
EXAMPLE 9.9

The state of plane stress at a point is shown on the element in Figure. Represent this state of stress on an element oriented 30° counterclockwise from the position shown.



EXAMPLE 9.9

The state of plane stress at a point is shown on the element in Figure. Represent this state of stress on an element oriented 30° counterclockwise from the position shown.

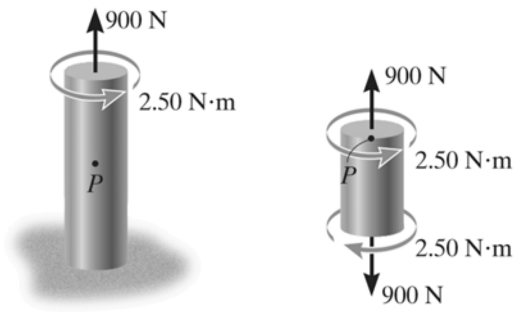


*9.5 Stress in Shafts Due to Axial Load and Torsion

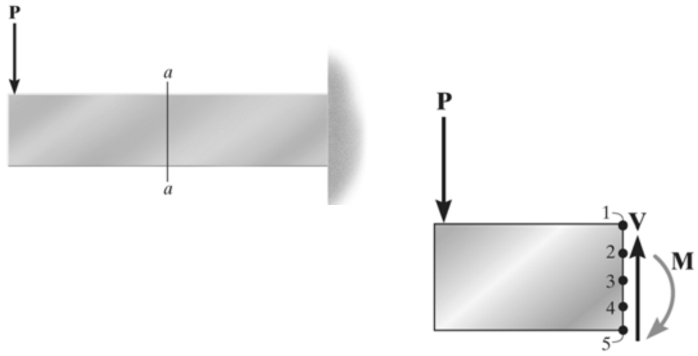
Occasionally circular shafts are subjected to the combined effects of both an axial load and torsion. Provided the material remains linear elastic, and is only subjected to small deformations, then we can use the principle of superposition to obtain the resultant stress in the shaft due to both of these loadings.

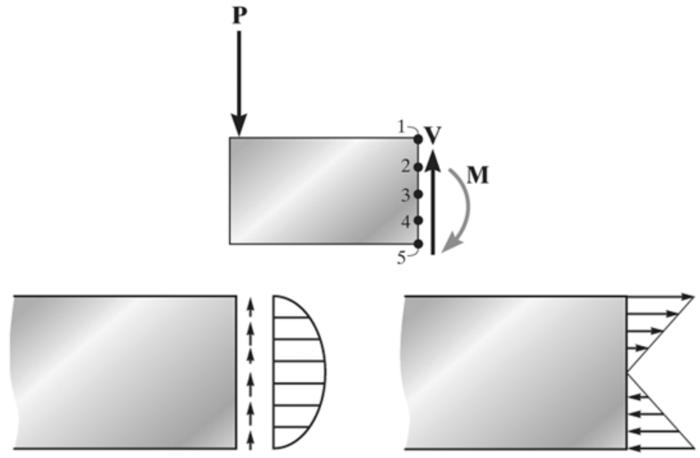
EXAMPLE 9.12

An axial force of 900 N and a torque of 2.50 N · m are applied to the shaft as shown in Figure. If the shaft has a diameter of 40 mm, determine the principal stresses at a point *P* on its surface.

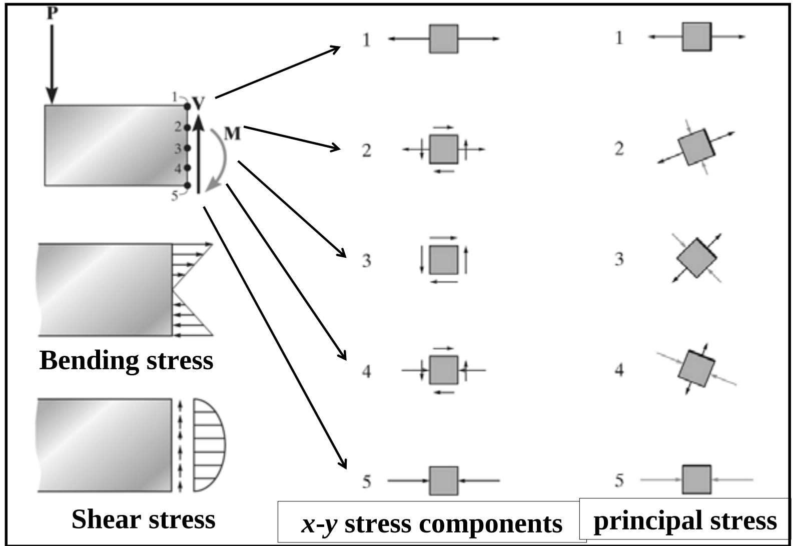


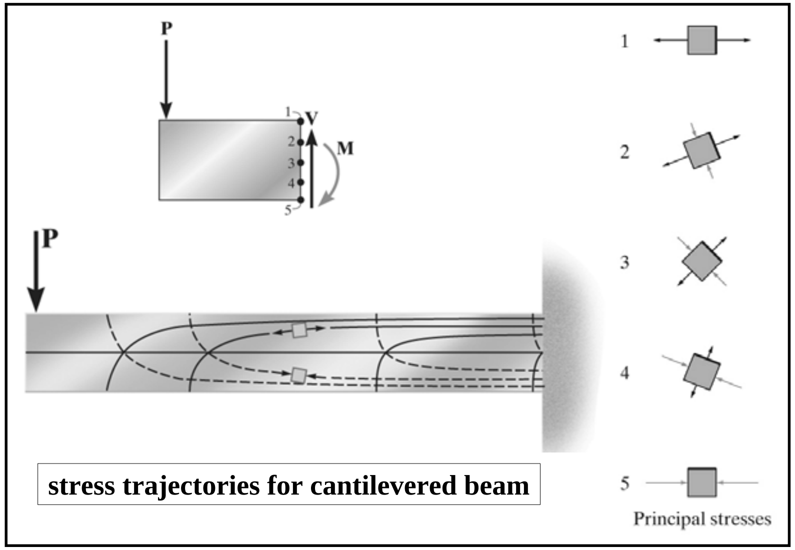
*9.6 Stress Variations Throughout a Prismatic Beam





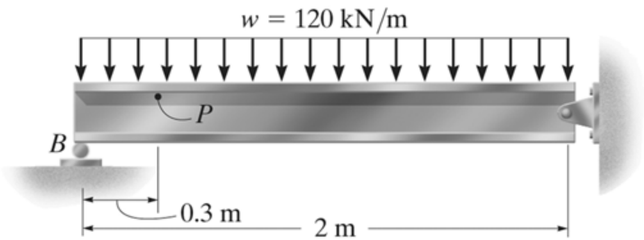
Shear stress distribution **Bending stress distribution**





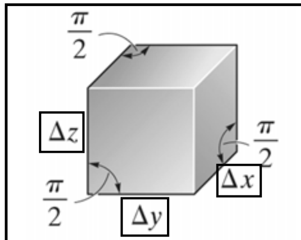
EXAMPLE 9.13

The beam shown in Figure is subjected to the distributed loading of w = 120 kN/m. Determine the principal stresses in the beam at point P, which lies at the top of the web. Neglect the size of the fillets and stress concentrations at this point. $I = 67.4(10^{-6})\ m^4$.



Chapter 10
Strain Transformation

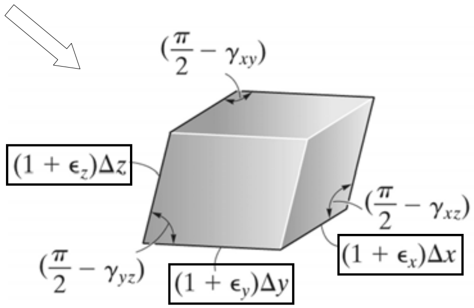
Complex stresses developed within this airplane wing are analyzed from strain gauge data.



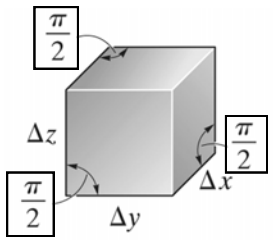
Undeformed element

✓ Normal strains cause a change in volume.

$$(1+\varepsilon_x)\Delta x \quad (1+\varepsilon_y)\Delta y \quad (1+\varepsilon_z)\Delta z$$



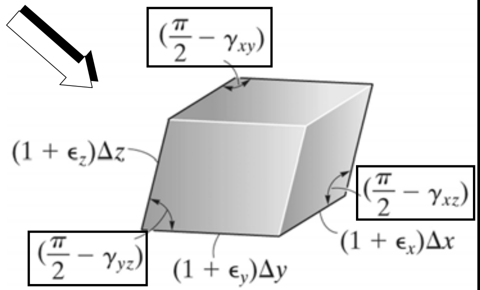
Deformed element



Undeformed element

✓ Shear strains cause a change in its shape.

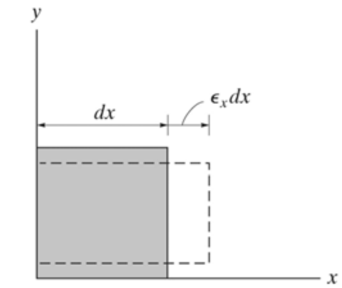
$$\frac{\pi}{2} - \gamma_{xy} \quad \frac{\pi}{2} - \gamma_{yz} \quad \frac{\pi}{2} - \gamma_{xz}$$



Deformed element

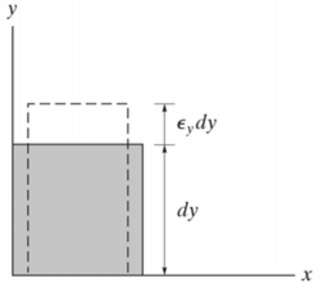
10.1 Plane-Strain

✓ A plain-strained element is subjected to two components of normal strain, $\varepsilon_x, \varepsilon_y$, and one component of shear strain, γ_{xy} .



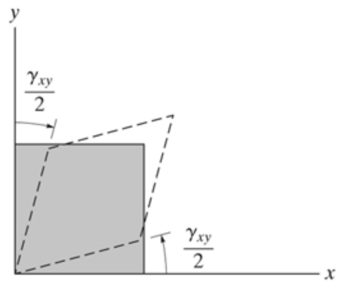
Normal strain ε_x

(change in length)



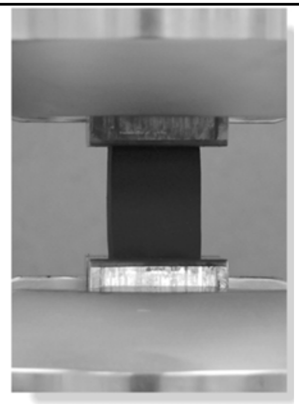
Normal strain ε_y

(change in length)

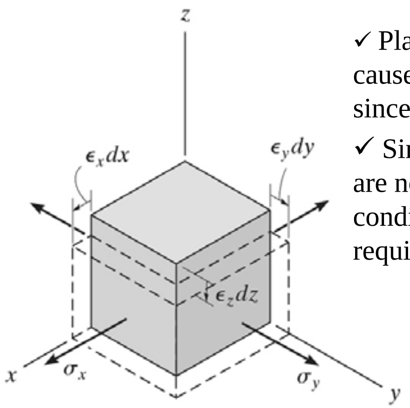


Shear strain γ_{xy}

(relative rotation of two adjacent sides of the element)



The rubber specimen is constrained between the two fixed supports, and so it will undergo plane strain when loads are applied to it in the horizontal plane.

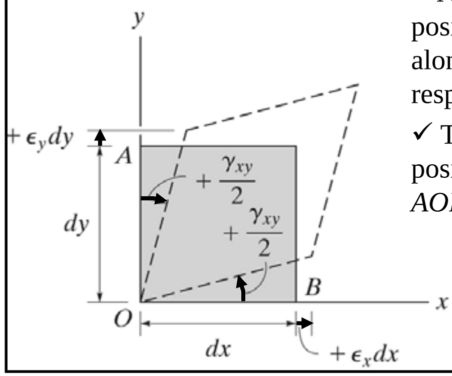


✓ Plane stress, σ_x, σ_y , does not cause plane strain in the *x*-*y* plane since $\varepsilon_z \neq 0$ (Poisson effect).

✓ Since shear stress and shear strain are not affected by Poisson's ratio, a condition of $\tau_{xz} = \tau_{yz} = 0$ requires $\gamma_{xz} = \gamma_{yz} = 0$.

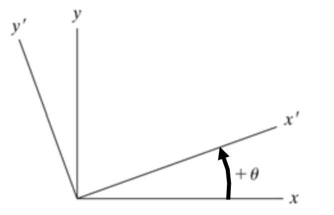
10.2 General Equations of Plane-Strain Transformation

○ **Sign Convention**

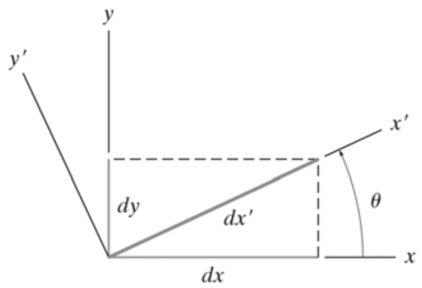


✓ Normal strains ε_x and ε_y are positive if they cause ***elongation*** along the *x* and *y* axes, respectively.

✓ The shear strain γ_{xy} is positive if the interior angle *AOB* becomes smaller than $90°$.



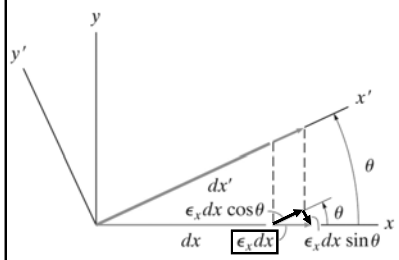
Positive sign convention



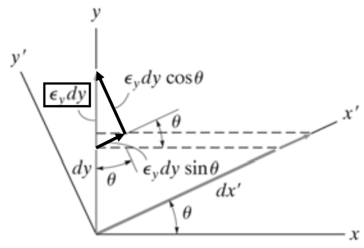
Before deformation

$$dx = dx' \cos\theta$$

$$dy = dx' \sin\theta$$



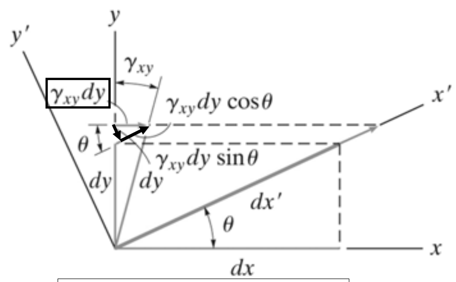
Normal strain ε_x



Normal strain ε_y

$$\delta x' = \varepsilon_x dx \cos\theta + \varepsilon_y dy \sin\theta$$

$$\delta y' = -\varepsilon_x dx \sin\theta + \varepsilon_y dy \cos\theta$$

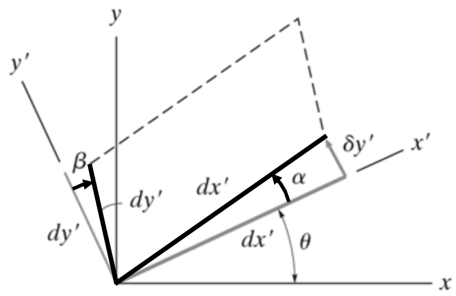


Shear strain γ_{xy}

$$\delta x' = \varepsilon_x dx \cos\theta + \varepsilon_y dy \sin\theta + \gamma_{xy} dy \cos\theta$$

$$\delta y' = -\varepsilon_x dx \sin\theta + \varepsilon_y dy \cos\theta - \gamma_{xy} dy \sin\theta$$

$$\boxed{\varepsilon_{x'} = \frac{\delta x'}{dx'} = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta}$$



$$\alpha = \delta y' / dx'$$

$$\alpha = \left(-\varepsilon_x + \varepsilon_y\right)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$$

Substituting $\theta + 90°$ for θ

$$\beta = \left(-\varepsilon_x + \varepsilon_y\right)\sin\left(\theta + 90°\right)\cos\left(\theta + 90°\right) - \gamma_{xy}\sin^2\left(\theta + 90°\right)$$

$$= -\left(-\varepsilon_x + \varepsilon_y\right)\cos\theta\sin\theta - \gamma_{xy}\cos^2\theta$$

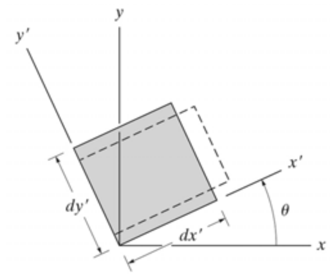
$$\gamma_{x'y'} = \alpha - \beta$$

$$= -2\left(-\varepsilon_x - \varepsilon_y\right)\sin\theta\cos\theta + \gamma_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

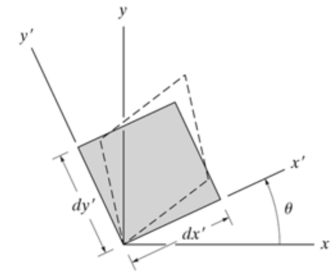
Using $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos^2\theta = \left(1+\cos 2\theta\right)/2$

$\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \quad \boxed{\begin{aligned} \varepsilon_{x'} &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \\ \frac{\gamma_{x'y'}}{2} &= -\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \end{aligned}}$$



Positive normal strain $\varepsilon_{x'}$



Positive shear strain $\gamma_{x'y'}$

Substituting $\theta + 90°$ for θ into $\varepsilon_{x'}$

$$\Rightarrow \quad \boxed{\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta}$$

Plane-stress transformation

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Plane-strain transformation

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

The similarity between two transformation equations

$$\sigma_x \rightarrow \varepsilon_x$$

$$\sigma_y \rightarrow \varepsilon_y$$

$$\sigma_{x'} \rightarrow \varepsilon_{x'}$$

$$\sigma_{y'} \rightarrow \varepsilon_{y'}$$

$$\tau_{xy} \rightarrow \frac{\gamma_{xy}}{2}$$

$$\tau_{x'y'} \rightarrow \frac{\gamma_{x'y'}}{2}$$

○ Principal Strains

✓ The element's deformation is represented by normal stains, with ***no shear strain***. When this occurs, the normal strains are referred to as ***principal strains***.

✓ If the material is **isotropic**, the axes along which these strains occur **coincide with** the axes that define the planes of principal stress.

$$\boxed{\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}}$$

$$\boxed{\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}}$$

○ Maximum In-Plane Shear Strain

$$\tan 2\theta_s = -\left(\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}\right)$$

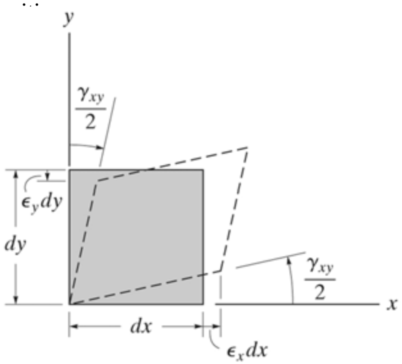
$$\frac{\gamma_{\substack{\max \\ in-plane}}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_{avg} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

Complex stresses are often developed at the joints where vessels are connected together. The stresses are determined by making measurements of strain.

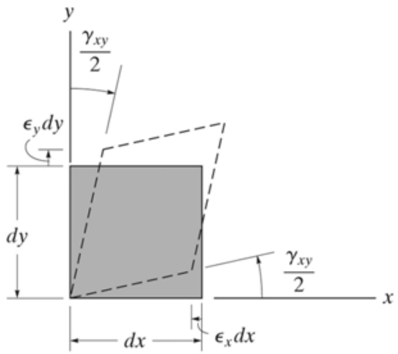
EXAMPLE 10.1

A differential element of material at a point is subjected to a state of plane strain $\varepsilon_x = 500(10^{-6})$, $\varepsilon_y = -300(10^{-6})$, $\gamma_{xy} = 200(10^{-6})$, which tends to distort the element as shown in Figure. Determine the equivalent strains acting on an element of the material oriented at the point, *clockwise* 30° from the original



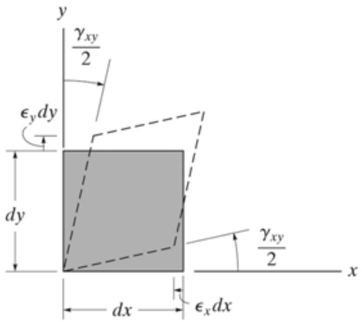
EXAMPLE 10.2

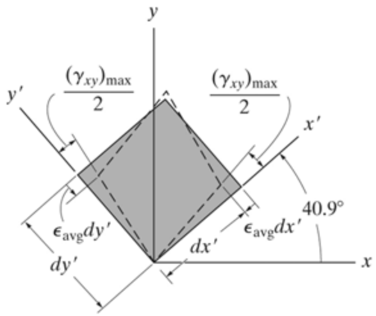
A differential element of material at a point is subjected to a state of plane strain defined by $\varepsilon_x = -350(10^{-6})$, $\varepsilon_y = 200(10^{-6})$, $\gamma_{xy} = 80(10^{-6})$, which tends to distort the element as shown in Figure. Determine the principal strains at the point and the associated orientation of the element.



EXAMPLE 10.3

A differential element of material at a point is subjected to a state of plane strain define by $\varepsilon_x = -350(10^{-6})$, $\varepsilon_y = 200(10^{-6})$, $\gamma_{xy} = 80(10^{-6})$, which tends to distort the element as shown in Figure. Determine the maximum in-plane shear strain at the point and the associated orientation of the element.



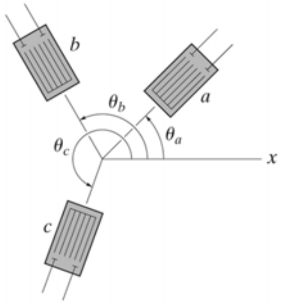


10.5 Strain Rosettes

✓ The normal strains at a point on its free surface are often determined using a cluster of three electrical-resistance strain gauges.

✓ Since these strains are measured only in the plane of the gauges, and the body is stress-free on its surface, the gauges may be subjected to plane stress but not plane strain.

✓ The out-of-plane displacement caused by principal normal strains in plane will not affect the in-plane measurements of the gauges.

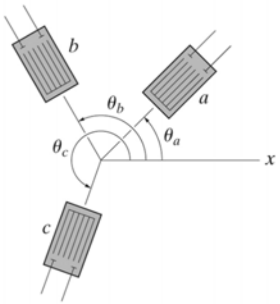


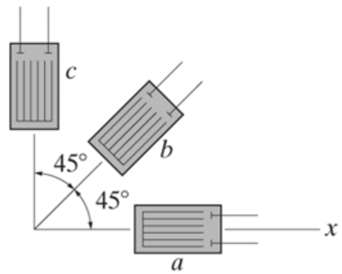
electrical-resistance strain gauge

$$\varepsilon_a = \varepsilon_x \cos^2\theta_a + \varepsilon_y \sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a$$
$$\varepsilon_b = \varepsilon_x \cos^2\theta_b + \varepsilon_y \sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b$$
$$\varepsilon_c = \varepsilon_x \cos^2\theta_c + \varepsilon_y \sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c$$

$$\Rightarrow \quad \varepsilon_x, \varepsilon_y, \gamma_{xy}$$

✓ Once $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ are determined, the transformation equations can be used to determine the principal in-plane strains and the maximum in-plane shear strain at the point.

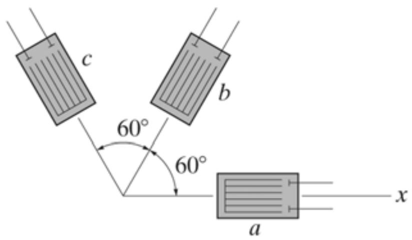




45° strain rosette

$$\theta_a = 0°,\ \theta_b = 45°,\ \theta_c = 90°$$

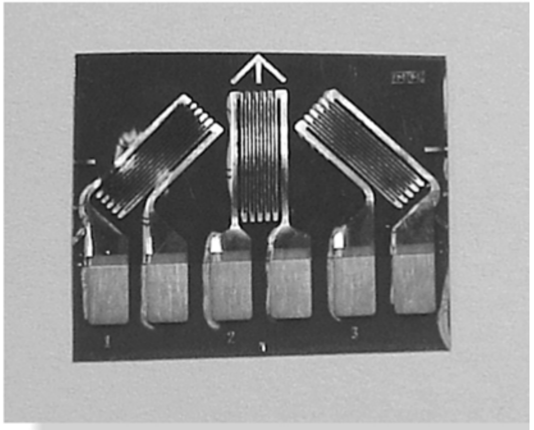
$$\varepsilon_x = \varepsilon_a$$
$$\varepsilon_y = \varepsilon_c$$
$$\gamma_{xy} = 2\varepsilon_b - (\varepsilon_a + \varepsilon_c)$$



60° strain rosette

$$\theta_a = 0°,\ \theta_b = 60°,\ \theta_c = 120°$$

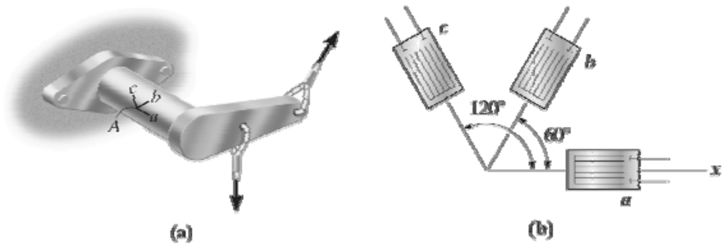
$$\varepsilon_x = \varepsilon_a$$
$$\varepsilon_y = \frac{1}{3}(2\varepsilon_b + 2\varepsilon_c - \varepsilon_a)$$
$$\gamma_{xy} = \frac{2}{\sqrt{3}}(\varepsilon_b - \varepsilon_c)$$



Typical electrical resistance $45°$ strain rosette.

EXAMPLE 10.8

The state of strain at point A on the bracket in Figure (a) is measured using the strain rosette shown in Figure (b). Due to the loadings, the reading from the gauges give $\varepsilon_a = 60(10^{-6})$, $\varepsilon_b = 135(10^{-6})$, $\varepsilon_c = 264(10^{-6})$. Determine the in-plane principal strains at the point and the directions in which they act.

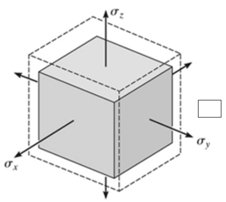


10.6 Material–Property Relationships

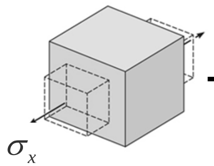
○ **Generalized Hooke's Law**

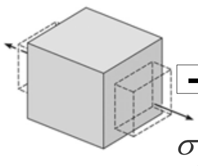
✓ Homogeneous and isotropic

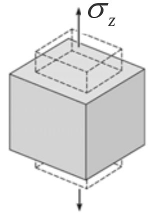
✓ Linear-elastic $\sigma = E\varepsilon$, $\varepsilon_{\text{lat}} = -\nu\varepsilon_{\text{long}}$



$\varepsilon'_x = \dfrac{\sigma_x}{E}$ $\qquad \varepsilon''_x = -\nu\dfrac{\sigma_y}{E}$ $\qquad \varepsilon'''_x = -\nu\dfrac{\sigma_z}{E}$

= 

+ 

+ 

By superposition

$$\varepsilon_x = \varepsilon'_x + \varepsilon''_x + \varepsilon'''_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

Generalized Hooke's Law for normal strains

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

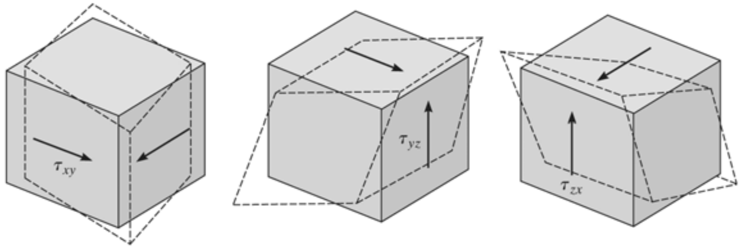
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

✓ Homogeneous and isotropic materials

✓ Linear-elastic behavior

✓ Small deformations



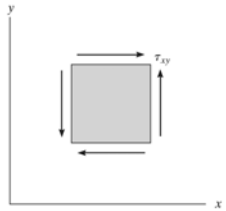
$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}$$

Generalized Hooke's Law for shear strains

○ Relationship Involving $E, \nu,$ and G

For pure shear $\sigma_x = \sigma_y = \sigma_z = 0$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0$$



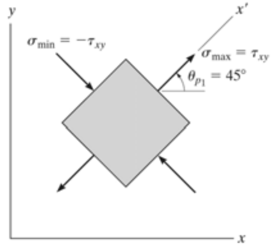
Recall $$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2}$$

Principal stresses

$$\sigma_{\max} = \tau_{xy} \quad \text{at} \quad \theta_{p1} = 45°$$

$$\sigma_{\min} = -\tau_{xy}$$



For $x' - y'$ axes

$$\varepsilon_{x'} = \varepsilon_{\max} = \frac{1}{E}\left[\sigma_{x'} - \nu\left(\sigma_{y'} + \sigma_{z'}\right)\right] = \frac{\tau_{xy}}{E}(1+\nu) \qquad (1)$$

Recall $$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

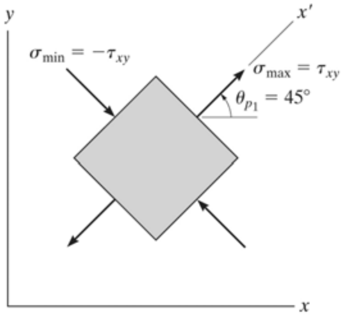
$$\because \varepsilon_x = \varepsilon_y = 0$$

$$\therefore \varepsilon_1 = \varepsilon_{\max} = \frac{\gamma_{xy}}{2} \qquad (2)$$

From (1) and (2)

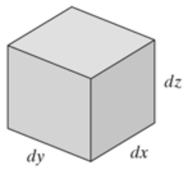
$$\frac{\tau_{xy}}{E}(1+\nu) = \frac{\gamma_{xy}}{2} = \frac{\tau_{xy}}{2G}$$

$$\Rightarrow \boxed{G = \frac{E}{2(1+\nu)}}$$



○ Dilatation and Bulk Modulus

$$\delta V = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\,dxdydz - dxdydz$$



Neglecting the products of the strains since the strains are very small.

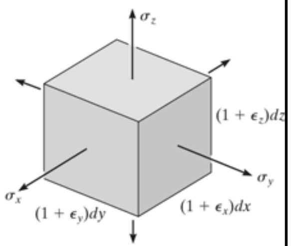
$$\delta V = \left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) dxdydz$$

Dilatation or *Volumetric strain*

$$e = \frac{\delta V}{dV} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Using the generalized Hooke's law

$$e = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$



✓ By comparison, the shear strains will not change the volume of the element, rather they will only change its rectangular shape

Hydrostatic loading

✓ When a volume element of material is subjected to the uniform pressure p of a liquid, the pressure on the body is the same in all directions and is always normal to any surface on which it acts.
✓ Shear stresses are *not present*, since the shear resistance of a liquid is zero.

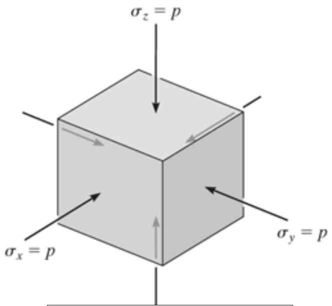
$$\sigma_x = \sigma_y = \sigma_z = -p$$

$$\rightarrow \frac{p}{e} = -\frac{E}{3(1-2\nu)}$$

Similar to Hooke's law $\sigma / \varepsilon = E$

$$\boxed{k = \frac{E}{3(1-2\nu)}}$$

***Bulk modulus* or**
volume modulus of elasticity



Hydrostatic stress

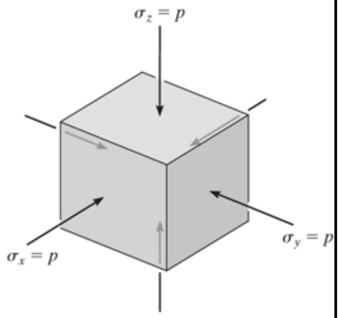
Bulk modulus

$$\boxed{k = \frac{E}{3(1-2\nu)}} \qquad k = \frac{p}{e}$$

✓ For most metals $\nu \approx \frac{1}{3}$ so $k \approx E$

✓ For rigid materials which did not change its volume $\delta V = 0$ so $k = \infty$

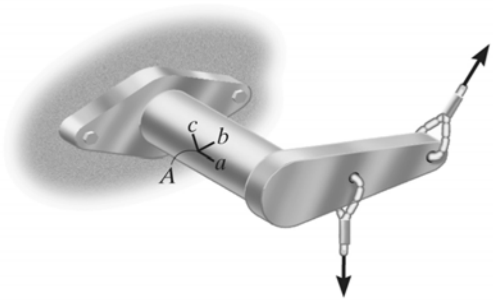
✓ The theoretical maximum value for Poisson's ratio $\nu = 0.5$. Also, during plastic yielding, no actual volume change of the material.



Hydrostatic stress

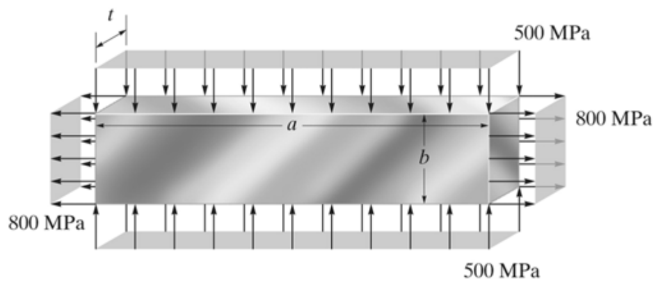
EXAMPLE 10.9

The bracket in Example 10-8 is made of steel for which E_{st} = 200 GPa, $\nu_{st} = 0.3$. Determine the principal stresses at point *A*.



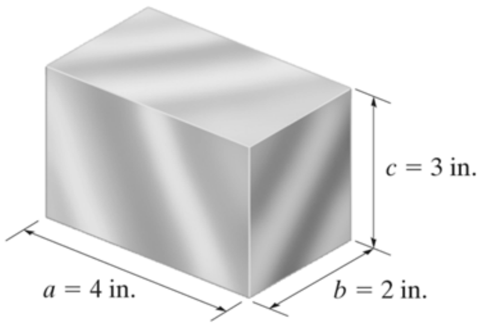
EXAMPLE 10.10

The copper bar in Figure is subjected to a uniform loading along its edges as shown. If it has a length a = 300 mm, width b = 50 mm, and thickness *t* = 20 mm before the load is applied, determine its new length, width, and thickness after application of the load. Take E_{cu} = 120 Gpa, $\nu = 0.34$.



EXAMPLE 10.11

If the rectangular block shown in Figure is subjected to a uniform pressure of p = 20 psi, determine the dilatation and the change in length of each side. Take E = 600 psi, $\nu = 0.45$.



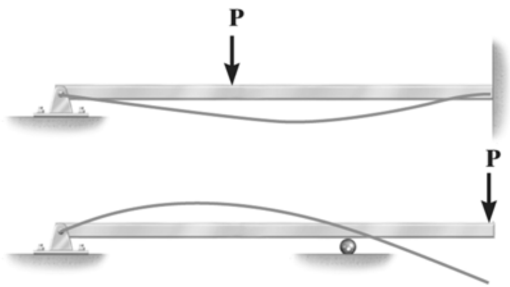
Chapter 12
Deflection of Beams and Shafts

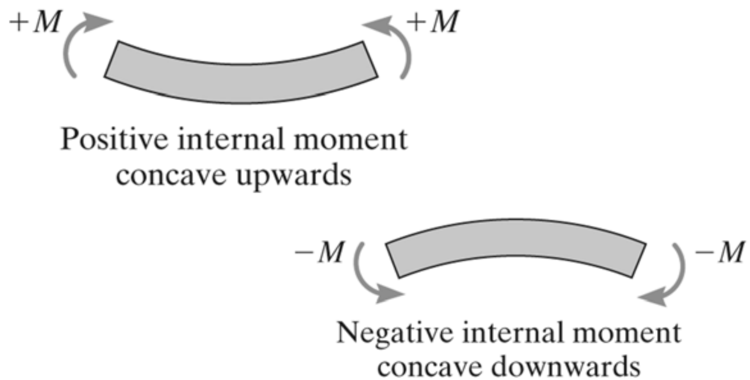
If the curvature of this pole is measure, it is then possible to determine the bending stress developed within it.

12.1 The Elastic Curve

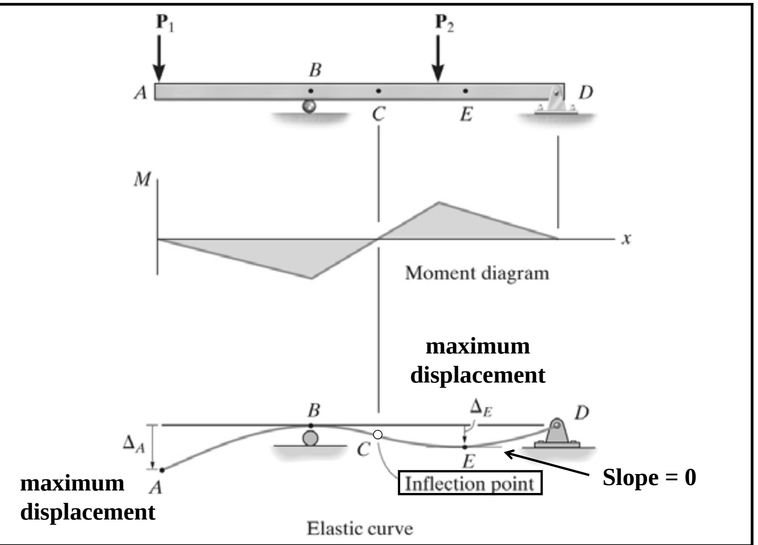
✓ The deflection diagram of the longitudinal axis that passes through the centroid of each cross-sectional area of the beam is called the ***elastic curve***.

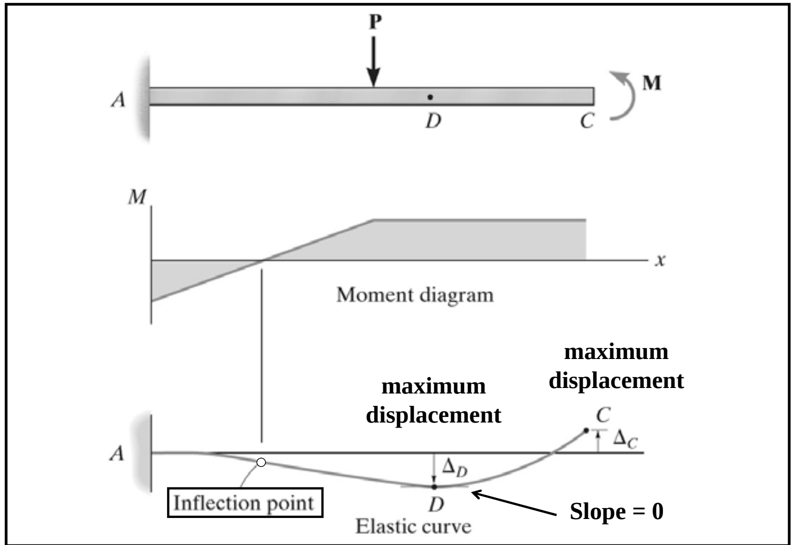
✓ It is necessary to know how the slop or displacement is restricted at various types of supports



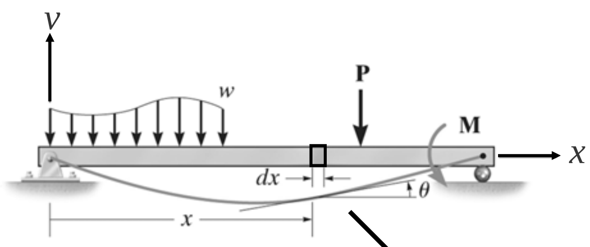


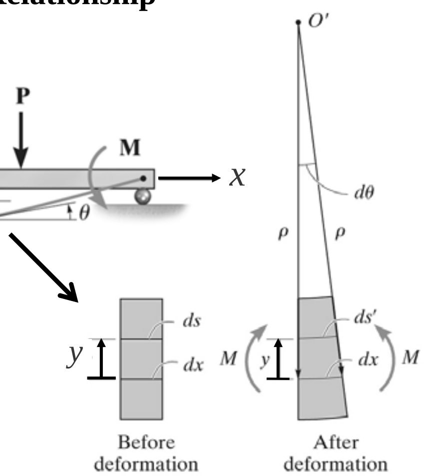
✓ If the moment diagram is known, it will be easy to construct the elastic curve.





○ Moment-Curvature Relationship





- ✓ straight beam
- ✓ symmetric cross section
- ✓ homogeneous
- ✓ linear-elastic

✓ Normal strain along ds $\quad \varepsilon = \dfrac{ds' - ds}{ds}$

$$ds = dx = \rho\, d\theta \qquad ds' = (\rho - y)d\theta$$

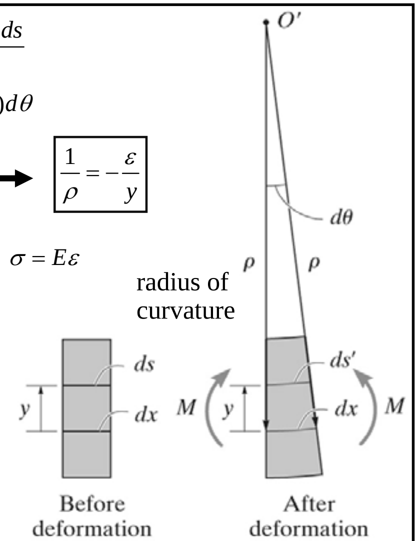
$$\varepsilon = \frac{(\rho - y)d\theta - \rho\, d\theta}{\rho\, d\theta} = -\frac{y}{\rho} \quad \rightarrow \quad \boxed{\frac{1}{\rho} = -\frac{\varepsilon}{y}}$$

✓ Homogeneous and linear-elastic $\quad \sigma = E\varepsilon$

✓ Flexure formula $\quad \sigma = -\dfrac{My}{I}$

$$\rightarrow \quad \boxed{\varepsilon = -\frac{My}{EI}}$$

$$\Rightarrow \quad \boxed{\frac{1}{\rho} = \frac{M}{EI}}$$



$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

Relationship between the internal moment and the radius of curvature

ρ : the radius of curvature at a specific point on the elastic curve
($1/\rho$ is referred to as the ***curvature*** κ)

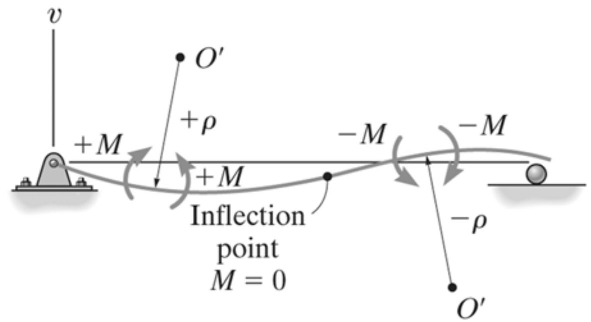
M : the resultant internal moment in the beam at the point where ρ is to be determined

E : the material's modulus of elasticity

I : the beam's moment of inertia of the cross-sectional area computed about the neutral axis

$$M = (EI)\kappa$$

EI = **Flexural rigidity**

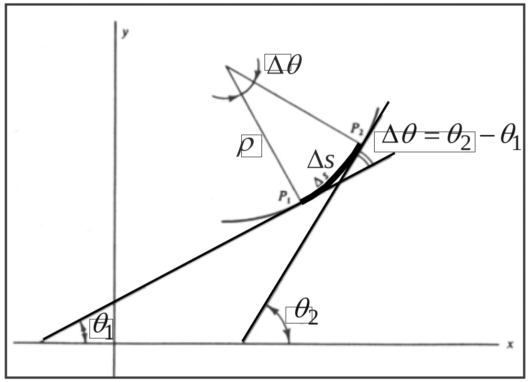


$$\frac{1}{\rho} = -\frac{\varepsilon}{y} \quad \longrightarrow \quad \boxed{\frac{1}{\rho} = -\frac{\sigma}{Ey}}$$

12.2 Slope and Displacement by Integration

The moment of inertia of this bridge girders vary along its length and this must be taken into account when computing its deflection.

Curvature of an Elastic Line

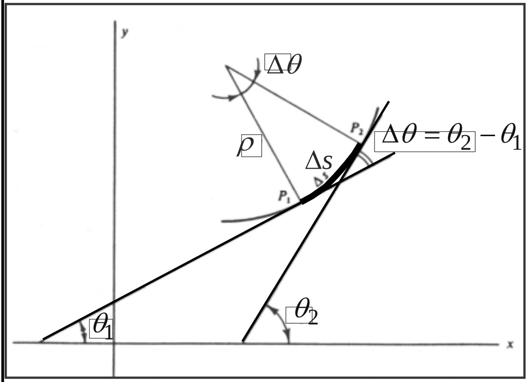


$$\kappa = \frac{1}{\rho}$$

$$= \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}$$

$$= \frac{d\theta}{dx}\frac{dx}{ds}$$

$$\kappa = \frac{d\theta}{dx}\frac{dx}{ds} \qquad \frac{d\theta}{dx} = ? \qquad \tan\theta = \frac{dy}{dx}$$

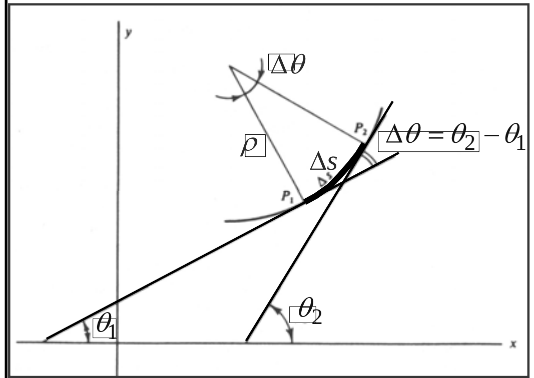


$$\frac{d}{dx}\tan\theta = \frac{d^2y}{dx^2}$$

$$(1+\tan^2\theta)\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d\theta}{dx} = \frac{d^2y/dx^2}{1+\left(dy/dx\right)^2}$$

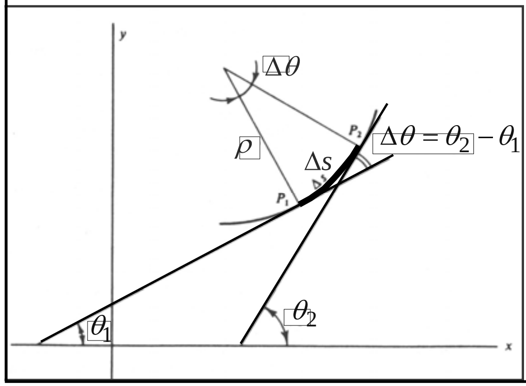
$$\kappa = \frac{d\theta}{dx}\frac{dx}{ds} \qquad \frac{dx}{ds} = ? \qquad ds^2 = dx^2 + dy^2$$



$$ds = (dx^2 + dy^2)^{1/2}$$

$$\frac{dx}{ds} = \frac{dx}{\left(dx^2+dy^2\right)^{1/2}} = \frac{1}{\left(\dfrac{dx^2+dy^2}{dx^2}\right)^{1/2}} = \frac{1}{\left[1+\left(dy/dx\right)^2\right]^{1/2}}$$

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx}\frac{dx}{ds} = \frac{d^2y/dx^2}{\left[1+\left(dy/dx\right)^2\right]^{3/2}}$$



Neglect $\left(dy/dx\right)^2$

$$\kappa = \frac{d\theta}{ds} \approx \frac{d^2y}{dx^2}$$

Elastic curve $v = f(x)$

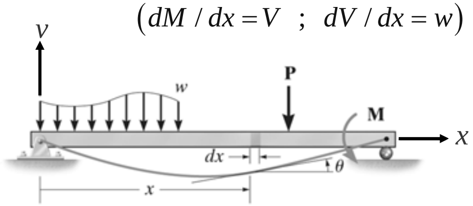
$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}} \quad \text{and} \quad \frac{1}{\rho} = \frac{M}{EI}$$

$$\Rightarrow \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}} = \frac{M}{EI}$$

nonlinear second-order differential equation

Assume dv/dx is very small $\Rightarrow$ $\dfrac{d^2v}{dx^2} = \dfrac{M}{EI}$

$$\left(dM/dx = V \;;\; dV/dx = w\right) \qquad \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x)$$



$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = w(x)$$

➢ Assume the flexural rigidity ***EI*** is constant along the length of the beam

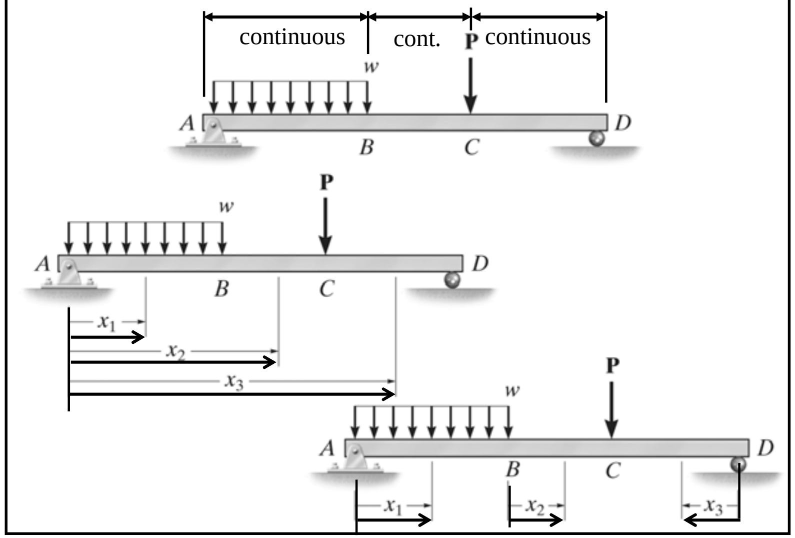
$$\boxed{\begin{aligned} EI\frac{d^4v}{dx^4} &= w(x) \\ EI\frac{d^3v}{dx^3} &= V(x) \\ EI\frac{d^2v}{dx^2} &= M(x) \end{aligned}}$$

Four constants of integration

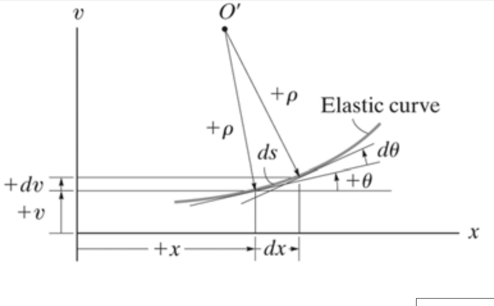
Three constants of integration

Two constants of integration

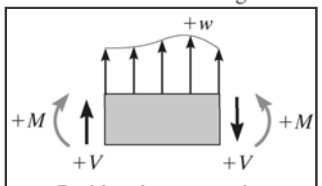
$$\frac{dv}{dx} = \tan\theta \approx \theta$$



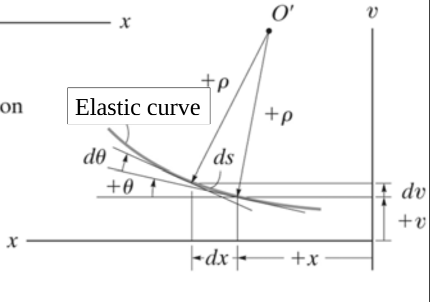
Sign Convention and Coordinates



Positive sign convention



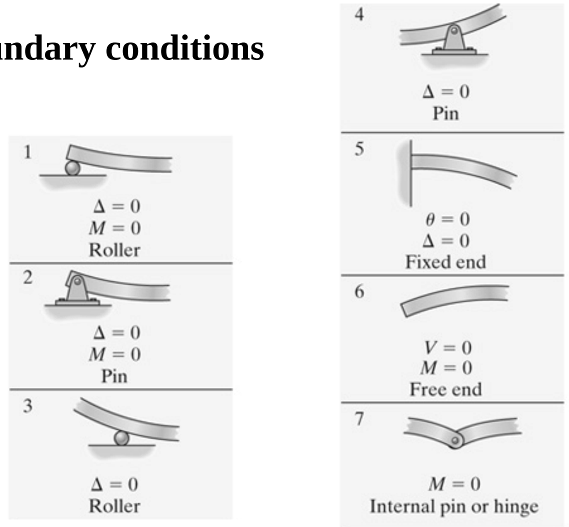
Positive sign convention



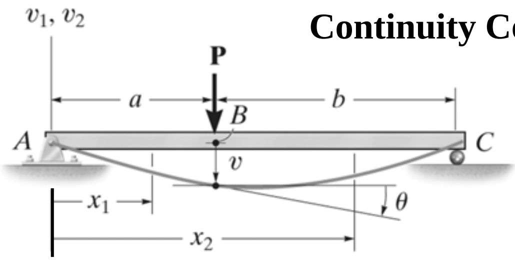
Positive sign convention

The design of a roof system requires a careful consideration of deflection. For example, rain can accumulate on areas of the roof, which then causes ponding, leading to further deflection, then further ponding, and finally possible failure of the roof.

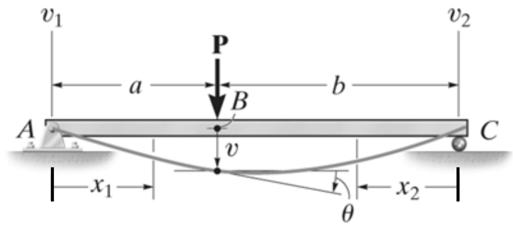
Boundary conditions



Continuity Conditions



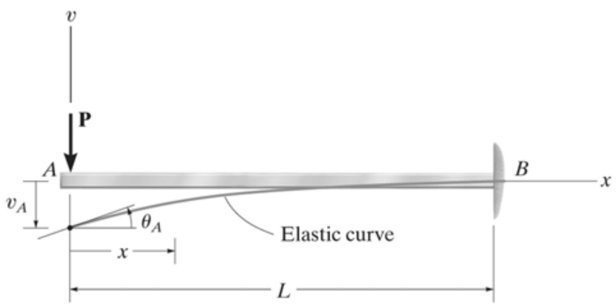
$$v_1(a) = v_2(a)$$
$$\theta_1(a) = \theta_2(a)$$



$$v_1(a) = v_2(b)$$
$$\theta_1(a) = -\theta_2(b)$$

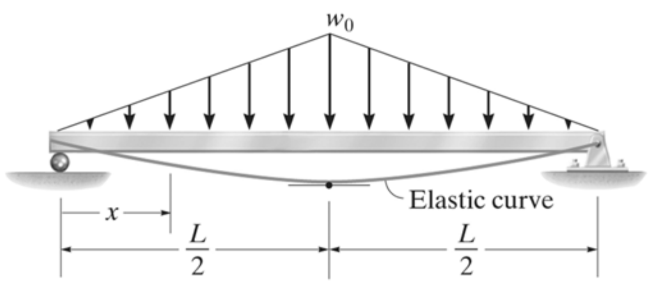
EXAMPLE 12.1

The cantilevered beam shown in Figure is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. *EI* is constant.



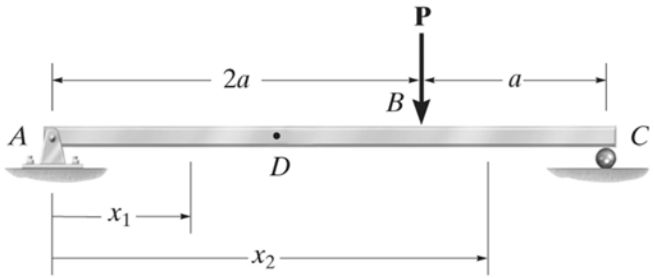
EXAMPLE 12.2

The simply supported beam shown in Figure supports the triangular distributed loading. Determine its maximum deflection. *EI* is constant.



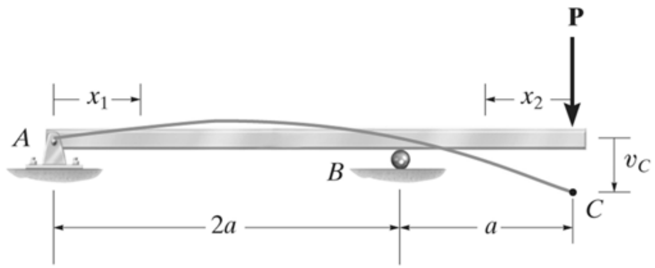
EXAMPLE 12.3

The simply supported beam shown in Figure is subjected to the concentrated force **P**. Determine the maximum deflection of the beam. *EI* is constant.



EXAMPLE 12.4

The beam in Figure is subjected to a load **P** at its end. Determine the displacement at *C*. *EI* is constant.



12.5 Method of Superposition

The resultant deflection at any point on this beam can be determined from the superposition of the deflections caused by each of the separate loadings acting on the beam.

$$EI\frac{d^4v}{dx^4} = w(x)$$

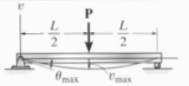
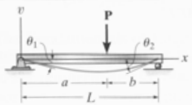
Two necessary requirements for applying the principle of superposition

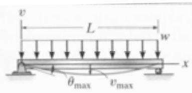
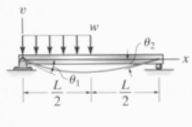
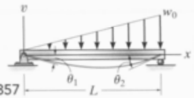
✓ The load $w(x)$ is linearly related to the deflection $v(x)$.

✓ The load is assumed not to change significantly the original geometry of the beam or shaft.

⇨ The deflection for a series of separate loadings acting on a beam may be superimposed.

Simply Supported Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{16EI}$	$v_{max} = \dfrac{-PL^3}{48EI}$	$v = \dfrac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \dfrac{-Pab(L+b)}{6EIL}$ $\theta_2 = \dfrac{Pab(L+a)}{6EIL}$	$v\Big\vert_{x=a} = \dfrac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \dfrac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \dfrac{-M_0L}{3EI}$ $\theta_2 = \dfrac{M_0L}{6EI}$	$v_{max} = \dfrac{-M_0L^2}{\sqrt{243}EI}$	$v = \dfrac{-M_0x}{6EIL}(x^2 - 3Lx + 2L^2)$

	$\theta_{max} = \dfrac{-wL^3}{24EI}$	$v_{max} = \dfrac{-5wL^4}{384EI}$	$v = \dfrac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \dfrac{-3wL^3}{128EI}$ $\theta_2 = \dfrac{7wL^3}{384EI}$	$v\Big\vert_{x=L/2} = \dfrac{-5wL^4}{768EI}$ $v_{max} = -0.006563\dfrac{wL^4}{EI}$ at $x = 0.4598L$	$v = \dfrac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \dfrac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \dfrac{-7w_0L^3}{360EI}$ $\theta_2 = \dfrac{w_0L^3}{45EI}$	$v_{max} = -0.00652\dfrac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \dfrac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

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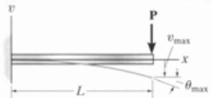
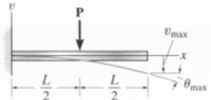
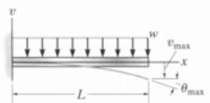
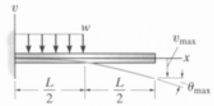
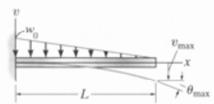
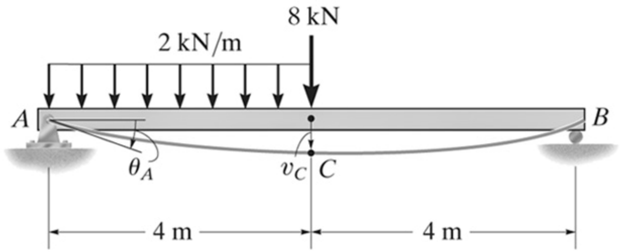
Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{2EI}$	$v_{max} = \dfrac{-PL^3}{3EI}$	$v = \dfrac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \dfrac{-PL^2}{8EI}$	$v_{max} = \dfrac{-5PL^3}{48EI}$	$v = \dfrac{-Px^2}{6EI}\left(\tfrac{3}{2}L - x\right)$ $0 \le x \le L/2$ $v = \dfrac{-PL^2}{24EI}\left(3x - \tfrac{1}{2}L\right)$ $L/2 \le x \le L$
	$\theta_{max} = \dfrac{-wL^3}{6EI}$	$v_{max} = \dfrac{-wL^4}{8EI}$	$v = \dfrac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$

	$\theta_{max} = \dfrac{M_0L}{EI}$	$v_{max} = \dfrac{M_0L^2}{2EI}$	$v = \dfrac{M_0x^2}{2EI}$
	$\theta_{max} = \dfrac{-wL^3}{48EI}$	$v_{max} = \dfrac{-7wL^4}{384EI}$	$v = \dfrac{-wx^2}{24EI}\left(x^2 - 2Lx + \tfrac{3}{2}L^2\right)$ $0 \le x \le L/2$ $v = \dfrac{-wL^3}{192EI}(4x - L/2)$ $L/2 \le x \le L$
	$\theta_{max} = \dfrac{-w_0L^3}{24EI}$	$v_{max} = \dfrac{-w_0L^4}{30EI}$	$v = \dfrac{-w_0x^2}{120EIL}(10L^3 - 10L^2x + 5Lx^2 - x^3)$

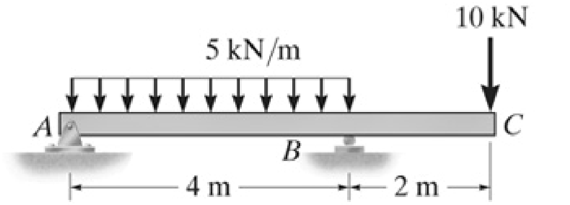
EXAMPLE 12.13

Determine the displacement at point *C* and the slope at the support *A* of the beam shown in figure. *EI* is constant.



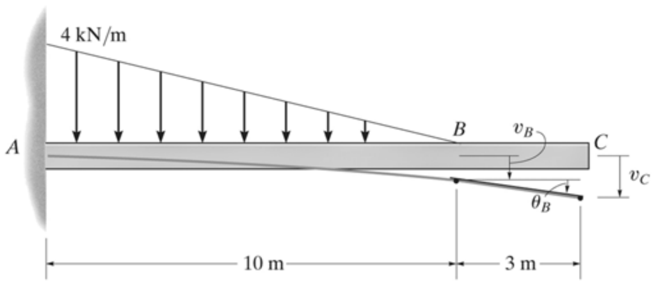
EXAMPLE 12.14

Determine the displacement at the end *C* of the overhanging beam shown in figure. *EI* is constant.



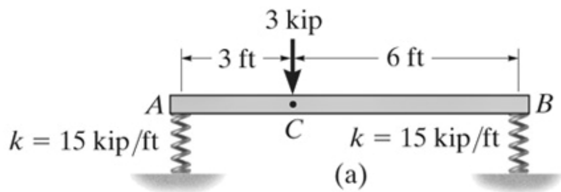
EXAMPLE 12.15

Determine the displacement at the end *C* of the cantilever beam shown in figure. *EI* is constant.



EXAMPLE 12.16

The steel bar shown in figure is supported by two springs at its ends *A* and *B*. Each spring has a stiffness of $k = 15$ kip/ft and is originally unstretched. If the bar is loaded with a force of 3 kips at point C, determine the vertical displacement of the force. Neglect the weight of the bar and take $E_{st} = 29000$ ksi, $I = 12$ in^4.



Chapter 13
Buckling of Columns

The columns used to support this water tank are braced at their mid-height in order to reduce their chance of buckling.

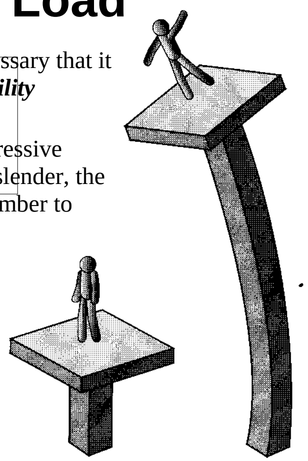
13.1 Critical Load

✓ Whenever a member is designed, it is necessary that it satisfy specific ***strength***, ***deflection***, and ***stability*** requirements.

✓ Some members may be subjected to compressive loadings, and if these members are long and slender, the loading may be large enough to cause the member to deflect laterally or sidesway.

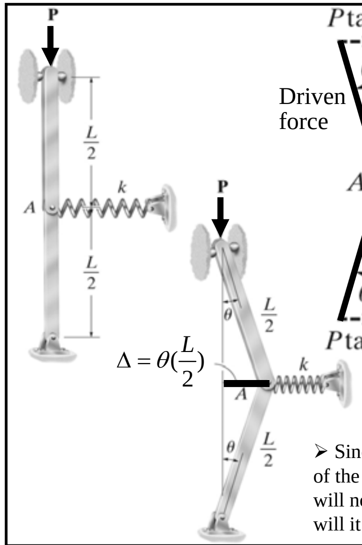
✓ Long slender members subjected to an axial compressive force are called ***columns***, and the lateral deflection that occurs is called ***buckling***.

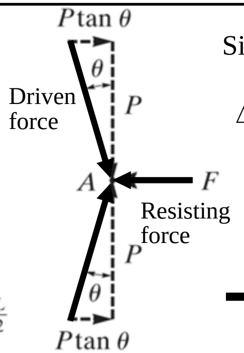
✓ Quite often the buckling of a column can lead to a sudden and dramatic failure of a structure or mechanism.



✓ The maximum axial load that a column can support which it is on the verge of buckling is called the ***critical load*** P_{cr} .

✓ Any additional loading will cause the column to buckle and therefore deflect laterally.





Since θ is small

$$\Delta \approx \theta(\frac{L}{2}) \qquad \tan\theta \approx \theta$$

$$\Rightarrow F = k\Delta = k\theta\frac{L}{2}$$

$$2P\tan\theta = 2P\theta = F$$

$$\Rightarrow 2P\theta = k\theta\frac{L}{2}$$

$$\boxed{P = \frac{kL}{4}}$$

➢ Since P_{cr} is independent of the (small) displacement θ of the bars, any slight disturbance given to the mechanism will not cause it to move further out of equilibrium, nor will it be restored to its original position.

$\left(2P\theta > k\theta\frac{L}{2}\right)$

driven force > resisting force

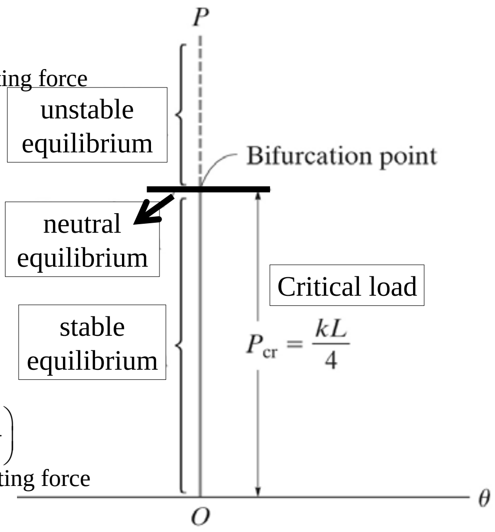
$P > \frac{kL}{4}$ unstable equilibrium

$P = \frac{kL}{4}$ neutral equilibrium

$P < \frac{kL}{4}$ stable equilibrium

$\left(2P\theta < k\theta\frac{L}{2}\right)$

driven force < resisting force



13.2 Ideal Column with Pin Supports

Some slender pin-connected members used in moving machinery, such as this short link, are subjected to compressive loads and thus act as columns

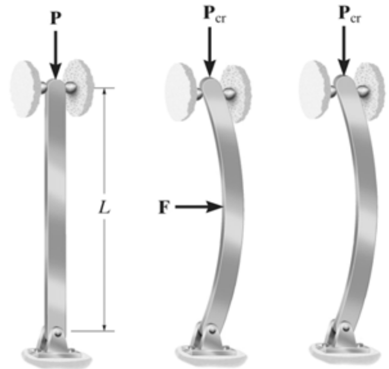
○ **Ideal Column**

✓ Perfectly straight before loading

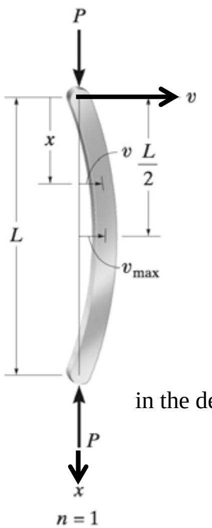
✓ Homogeneous material

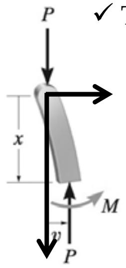
✓ The load is applied through the centroid of the cross section.

✓ The column buckles or bends in a single plane.



➢ A small lateral force will cause the column to remain in the deflected position when F is removed.





in the deflected position

Assumptions

✓ The slope of the elastic curve is small.

✓ The deflections occur only by bending.

$$EI\frac{d^2v}{dx^2} = M$$

By using the method of sections

$$M = -Pv$$

$$EI\frac{d^2v}{dx^2} = -Pv$$

$$\frac{d^2v}{dx^2} + \left(\frac{P}{EI}\right)v = 0$$

➢ A homogeneous, second-order, linear differential equation with constant coefficients

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$$\frac{d^2v}{dx^2} + \left(\frac{P}{EI}\right)v = 0$$

The general solution

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}\ x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}\ x\right)$$

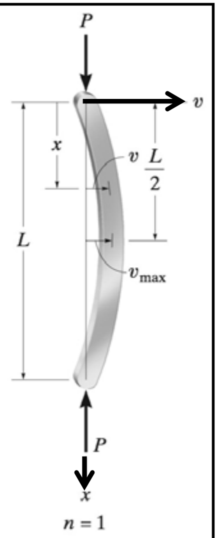
Boundary conditions

(1) $x = 0,\ \ v = 0$ ➡ $C_2 = 0$

(2) $x = L,\ \ v = 0$ ➡ $C_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$

If $C_1 = 0$, then $v = 0$ **trivial solution**

If $C_1 \neq 0$, $\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$



$$\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0 \quad ➡ \quad \sqrt{\frac{P}{EI}}L = n\pi$$

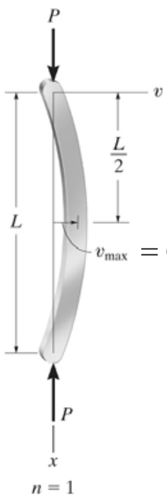
$$➡ \quad P = \frac{n^2\pi^2 EI}{L^2} \qquad n = 1, 2, 3, \ldots$$

The smallest value of *P* (when $n = 1$)

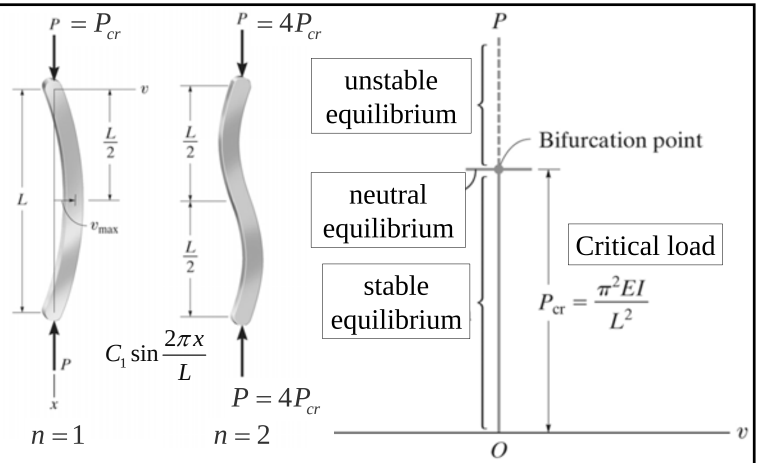
$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad \textbf{\textit{Euler load}}$$

The corresponding buckled shape

$$v = C_1 \sin\frac{\pi x}{L}$$



$= C_1$



➢ The critical load is independent of the strength of the material; rather it depends only on the column's dimensions (*I* and *L*) and the material's stiffness or modulus of elasticity *E*.

➢ A column will buckle about the principal axis of the cross section having the ***least moment of inertia*** (the weakest axis).

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\because \; I_{a-a} < I_{b-b}$$

$$\therefore \; (P_{cr})_{a-a} < (P_{cr})_{b-b}$$

➡ The column will buckle about the *a-a* axis.

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$$

The buckling equation for a pin-supported long slender column

P_{cr} : critical or maximum axial load on the column just before it begins to buckle. This load must not cause the stress in the column to exceed the proportional limit

E : modulus of elasticity for the material

I : least moment of inertia for the column's cross-sectional area

L : unsupported length of the column, whose ends are pinned

For the purpose of design

$$I = A\,r^2 \;\rightarrow\; P_{cr} = \frac{\pi^2 E(Ar^2)}{L^2} \;\rightarrow\; \left(\frac{P}{A}\right)_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

r : ***radius of gyration***

$$\boxed{\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}}$$

L/r **= slenderness ratio**
(geometric ratio)

σ_{cr} : critical stress, which is an average stress in the column just before the column buckles. This stress is an ***elastic stress*** and therefore $\sigma_{cr} < \sigma_Y$

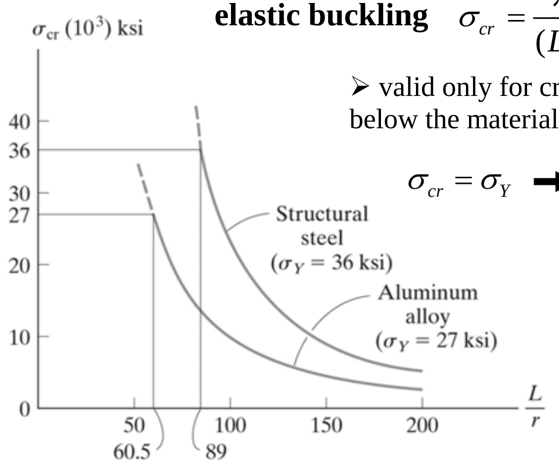
E : modulus of elasticity for the material

L : unsupported length of the column, whose ends are pinned

r : smallest radius of gyration of the column, determined from $r = \sqrt{I/A}$, where I is the least moment of inertia of the column's cross-sectional area A

Typical interior steel pipe columns used to support the roof of a single story building.

elastic buckling $\sigma_{cr} = \dfrac{\pi^2 E}{(L/r)^2}$

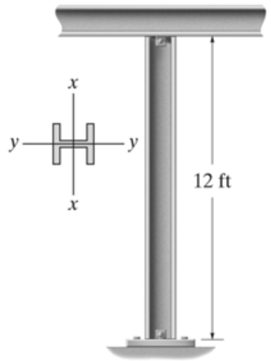


➢ valid only for critical stresses below the material's yield point

$$\sigma_{cr} = \sigma_Y \;\rightarrow\; (L/r)_{smallest}$$

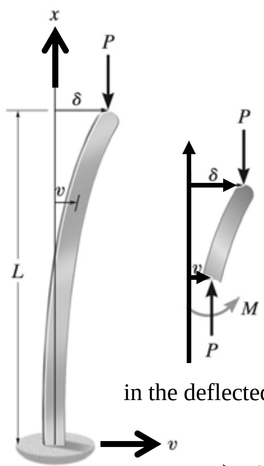
EXAMPLE 13.1

The A-36 steel $W8 \times 31$ member shown in figure is to be used as a pin-connected column. Determine the largest axial load it can support before it either begins to buckle or the steel yields.



13.3 Columns Having Various Types of Supports

The tubular columns used to support this water tank have been braced at three locations along their length to prevent them from buckling.



in the deflected position

Assumptions

✓ The slope of the elastic curve is small.

✓ The deflections occur only by bending.

$$EI\frac{d^2v}{dx^2} = M$$

By using the method of sections

$$M = P(\delta - v)$$

$$EI\frac{d^2v}{dx^2} = P(\delta - v)$$

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = \frac{P}{EI}\delta$$

➢ A nonhomogeneous, second-order, linear differential equation with constant coefficients

$$\frac{d^2v}{dx^2}+\frac{P}{EI}v=\frac{P}{EI}\delta$$

The solution consists of both a complementary and particular solution.

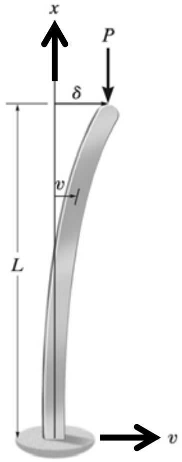
$$v=C_1\sin\left(\sqrt{\frac{P}{EI}}x\right)+C_2\cos\left(\sqrt{\frac{P}{EI}}x\right)+\delta$$

Boundary conditions

(1) $x=0,\quad v=0 \;\rightarrow\; C_2=-\delta$

$$\frac{dv}{dx}=C_1\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}x\right)-C_2\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}x\right)$$

(2) $x=0,\quad dv/dx=0 \;\rightarrow\; C_1=0$



The deflection curve

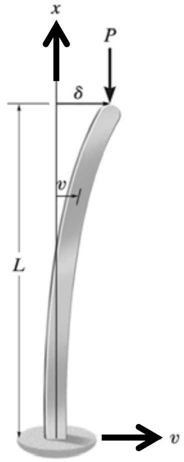
$$v=\delta\left[1-\cos\left(\sqrt{\frac{P}{EI}}x\right)\right]$$

(3) $x=L,\quad v=\delta$

$$\delta\cos\left(\sqrt{\frac{P}{EI}}L\right)=0$$

If $\delta=0$, then $v=0$ **trivial solution**

$$\rightarrow \cos\left(\sqrt{\frac{P}{EI}}L\right)=0 \;\rightarrow\; \sqrt{\frac{P}{EI}}L=\frac{n\pi}{2}$$



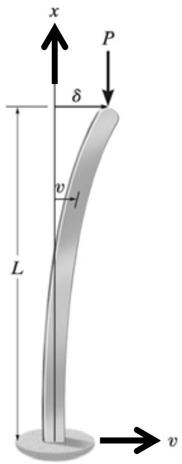
$$\rightarrow P=\frac{n^2\pi^2EI}{4L^2}\qquad n=1,3,5,\ldots$$

$$v=\delta\left[1-\cos\frac{n\pi x}{2L}\right]$$

The smallest value of *P* (when $n=1$)

$$P_{cr}=\frac{\pi^2EI}{4L^2}$$

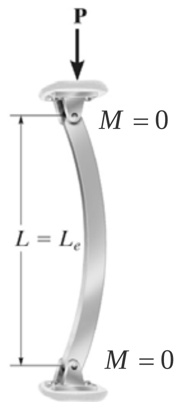
$$v=\delta\left[1-\cos\frac{\pi x}{2L}\right]$$



○ Effective Length

Euler formula $P_{cr}=\dfrac{\pi^2EI}{L^2}$

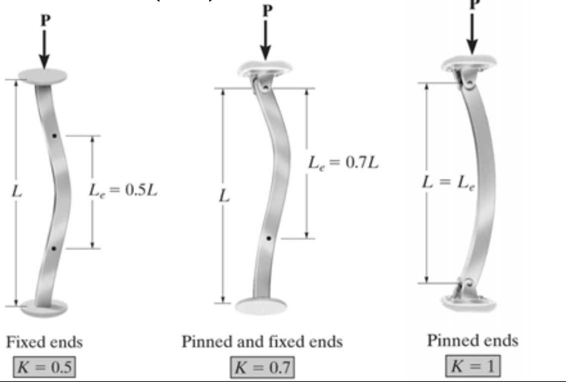
✓ L is the unsupported distance between the point of zero moment.
✓ This distance between the zero-moment points is called the column's ***effective length*** L_e .

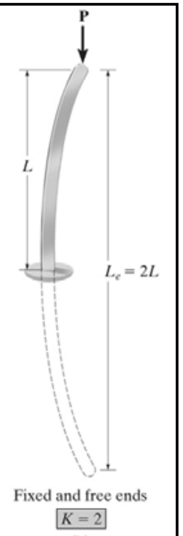


Pinned ends
$K=1$

Effective-length factor *K* $L_e = KL$

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$





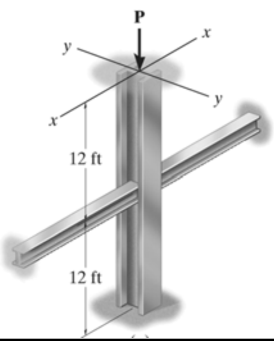
Euler formula

$$\boxed{P_{cr} = \frac{\pi^2 EI}{(KL)^2}} \quad \textbf{or} \quad \boxed{\sigma_{cr} = \frac{\pi^2 E}{(KL/r)^2}}$$

Effective slenderness ratio: (*KL*/*r*)

EXAMPLE 13.2

A W6×15 steel column is 24 ft long and is fixed at its ends as shown in Figure. Its load-carrying capacity is increased by bracing it about the y-y (weak) axis using struts that are assumed to be pin-connected to its midheight. Determine the load it can support so that the column does not buckle nor the material exceed the yield stress. Take $E_{st} = 29(10^3)$ ksi and $\sigma_Y = 60$ ksi.



EXAMPLE 13.3

The aluminum column is fixed at its bottom and is braced at its top by cables so as to prevent movement at the top along the *x* axis as shown in figure. If it is assumed to be fixed at its base, determine the largest allowable load *P* that can be applied. Use a factor of safety for buckling of F.S.=3.0. Take $E_{al} = 70$ GPa, $\sigma_Y = 215$ MPa, $A = 7.5(10^{-3})\ \text{m}^2$, $I_x = 61.3(10^{-6})\ \text{m}^4$, $I_y = 23.2(10^{-6})\ \text{m}^4$.